Numerical Solution of a Fractional Model for HIV Infection of CD4$^+$T Cells via Legendre Multiwavelet Functions

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Received: January 31, 2019 **Accepted: February 18, 2020**

Published: December 31, 2020

Abstract: *In this paper, we introduce a Galerkin method based on Legendre multiwavelet functions to obtain approximate solutions of a fractional model for HIV infection of CD4$^+$T cells corresponding to a class of systems of nonlinear fractional differential equations. The method converts the model problem into a system of nonlinear algebraic equations. A numerical example is included to demonstrate the validity and applicability of the technique and the results are compared with those obtained by existing methods.*

Keywords: *Galerkin method, Legendre multiwavelet functions, Fractional model for HIV infection of CD4$^+$T cells.*

Introduction

Differential equations are used to model various practical problems. In this article, we shall study the following system of equations:

$$\begin{cases} D^{\alpha_1}(T) = q - \alpha T + rT\left(1 - \frac{T+I}{T_{max}}\right) - kVT, \\ D^{\alpha_2}(I) = kVT - \beta I, \\ D^{\alpha_3}(V) = \mu\beta I - \gamma V, \end{cases} \quad T(0) = T_0,\ I(0) = I_0,\ V(0) = V_0,\ 0 \le t \le R < \infty, \tag{1}$$

where $0 < \alpha_i \le 1$, $i = 1,2,3$. Here D^{α_i} denotes the fractional derivative in the Caputo sense (see Definition 2). These equations describe the fractional model for HIV infection of CD4$^+$T cells [13]. The number of CD4$^+$T cells for a healthy person is $\frac{800}{1200}$ mm^3. In the system (1) R is any positive constant, $T(t)$, $I(t)$ and $V(t)$ respectively denote the concentration of susceptible CD4$^+$T cells, the number of CD4$^+$T cells infected by the HIV viruses and the free HIV virus partials in the blood at time t. α, β and γ denote the natural turnover rates of uninfected T cells, infected T cells and virus partials, respectively. The term $\left(1 - \frac{T+I}{T_{max}}\right)$ describes the logistic growth of the healthy CD4$^+$T cells and kVT describes the incidence of HIV infection of healthy CD4$^+$T cells, where $k > 0$ is the infection rate. It is assumed that each infected CD4$^+$T cell together with its daughter cell produces μ virus particles during their combined

lifetimes. The body is believed to produce CD4$^+$T cells from precursors in the bone marrow and thymus at constant rate q. T cells multiply through mitosis with a rate r when T cells are stimulated by antigen or mitogen. T_{max} denotes the maximum CD4$^+$T cells concentration in blood [1, 11, 14, 17].

The theory of wavelets is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines and, in particular, have been very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis, and fast algorithms for easy implementation [3]. Wavelets analysis has many useful properties, such as orthogonality, compact support, exact representation of polynomials to a given degree, and the ability to represent functions at different levels of resolution [10]. Moreover, wavelets can be used to construct fast numerical algorithms [2].

In this paper, we introduce a method, based on Legendre multiwavelet functions for solving system (1). These set of equations have already been solved by various numerical methods such as the Laplace Adomian decomposition method (LADM) [12], the homotopy perturbation method [8], the Pade approximate and the modified variational iteration method [9] and the Bessel collocation method [19]. Gandomani and Kajani [5] used the collocation method based on the Müntz-Legendre polynomials to solve system (1) and Gökdoĝan et al. [6] developed a multi-step differential transform method to give approximate as well as analytical solutions.

The article is organized as follows. In Section 2, we first describe the basic definitions of fractional calculus theory and then give the basic definition of Legendre multiwavelet functions and state their properties. In Section 3, we discuss approximations to functions using Legendre multiwavelet functions basis. Section 4 is devoted to describe a Galerkin method for solutions of system (1) based on the Legendre multiwavelet functions. In Section 5, we report numerical results to demonstrate the accuracy of the present technique. These results are compared with existing methods. Finally, Section 6 contains a conclusion.

Preliminaries

Basic definitions of fractional calculus

In this section, we present basic definitions in fractional calculus theory which will be needed later [4, 7].

Definition 1. The Riemann-Liouville fractional integral of order α for $f \in L_1[a,b]$, $t > 0$, $\alpha, t \in \mathbb{R}$, is defined by

$$J^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases}$$

Here $\Gamma(\mu)$ is the gamma function:

$$\Gamma(\mu) = \int_0^{\infty} e^{-s} s^{\mu-1} ds.$$

We have:

$$J^0 f(t) = f(t),$$

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t),$$

$J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t),$

where $f \in L_1[a,b]$, $\alpha, \beta \geq 0$.

The Riemann-Liouville fractional derivative of order α for $n-1 < \alpha \leq n \in \mathbb{N}$ is defined by

$$\mathscr{D}^{\alpha}f(t) = \frac{d^n}{dt^n}\left(J^{n-\alpha}f(t)\right).$$

Definition 2. The Caputo fractional derivative of order α for $f \in L_1[a,b]$, $t > 0$, $\alpha, t \in \mathbb{R}$ is defined as

$$D^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, & n-1 < \alpha < n, \\ \frac{d^n}{dt^n}f(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

Caputo's differential operator coincides with the usual differential operator of an integer order and has the following properties:

$D^{\alpha}J^{\alpha}f(t) = f(t),$

$$J^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{(t-a)^k}{k!}.$$

Legendre multiwavelet functions

The Legendre multiwavelet functions on interval $[0,T)$ are defined by [15, 18]

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2m+1}\,\frac{2^{\frac{l}{2}}}{\sqrt{T}}p_m\left(\frac{2^l t}{T} - n\right), & \frac{nT}{2^l} \leq t < \frac{(n+1)T}{2^l}, \\ 0, & otherwise, \end{cases}$$

where $m = 0,1,...,M-1$, $n = 0,1,...,2^l-1$, l can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. $\{\psi_{n,m}(t)\}$ is an orthonormal set. The coefficient $\sqrt{2m+1}$ is needed for orthonormality.

Legendre polynomials on the interval $[0,1]$ can be determined using the following recursive formula:

$p_0(t) = 1,$

$p_1(t) = t,$

$$p_{m+1}(t) = \frac{2m+1}{m+1}tp_m(t) - \frac{m}{m+1}p_{m-1}(t), \ m = 1,2,....$$

Function approximation

Consider a function $f(t)$ defined over $[0,T)$. We can approximate $f(t)$ as follows:

$$f(t) = \sum_{n=0}^{2^l-1}\sum_{m=0}^{\infty} a_{n,m}\psi_{n,m}(t), \tag{2}$$

where

$$a_{n,m} = \langle f, \psi_{n,m} \rangle, \;\; n = 0,1,...,2^l - 1, \;\; m \in \mathbb{N} \cup \{0\} \tag{3}$$

and l is any positive integer.

The convergence of the Chebyshev wavelet method is presented in [16]. We have the following convergence for Legendre multiwavelet approximations.

Theorem 1. Let $\sum_{n=0}^{2^l-1} \sum_{m=0}^{\infty} a_{n,m} \psi_{n,m}(t)$ be the Legendre multiwavelet approximation of $f \in H = L^2([0,T])$ and l be any fix positive integer, then $f_M(t) = \sum_{n=0}^{2^l-1} \sum_{m=0}^{M-1} a_{n,m} \psi_{n,m}(t)$ convergences to $f(t)$ as $M \to \infty$.

Proof. We will prove that the sequence of partial sums of $f_M(t)$, is a Cauchy sequence in Hilbert space of H. Let $f_N(t)$ be arbitrary partial sums of $f(t)$, and $M > N$. Then we have

$$\begin{aligned}
\| f_M(t) - f_N(t) \|_2^2 &= \| \sum_{n=0}^{2^l-1} \sum_{m=N}^{M-1} a_{n,m} \psi_{n,m}(t) \|_2^2 \\
&= \left\langle \sum_{n=0}^{2^l-1} \sum_{m=N}^{M-1} a_{n,m} \psi_{n,m}(t), \sum_{s=0}^{2^l-1} \sum_{r=N}^{M-1} a_{s,r} \psi_{s,r}(t) \right\rangle \\
&= \sum_{n=0}^{2^l-1} \sum_{m=N}^{M-1} \sum_{s=0}^{2^l-1} \sum_{r=N}^{M-1} a_{n,m} \overline{a_{s,r}} \langle \psi_{n,m}(t), \psi_{s,r}(t) \rangle \\
&= \sum_{n=0}^{2^l-1} \sum_{m=N}^{M-1} | \, a_{n,m} \, |^2 .
\end{aligned}$$

Since $\sum_{n=0}^{2^l-1} \sum_{m=0}^{\infty} | \, a_{n,m} \, |^2$ is a monotone series and bounded by $\|f\|_2^2$, it converges and hence its partial sums form a Cauchy sequence. Thus, $\sum_{n=0}^{2^l-1} \sum_{m=N}^{M-1} | \, a_{n,m} \, |^2$ converges to zero as $M,N \to \infty$. So $\| f_M(t) - f_N(t) \|_2^2$ converges to zero as $M,N \to \infty$. Thus, $f_M(t)$ is a Cauchy sequence and hence $f_M(t)$ converges to $g \in H$.

Now we show that $g(t) = f(t)$. By Eq. (3), we have

$$\begin{aligned}
\langle f(t) - g(t), \psi_{n,m}(t) \rangle &= \langle f(t), \psi_{n,m}(t) \rangle - \langle g(t), \psi_{n,m}(t) \rangle \\
&= \lim_{M \to \infty} \langle f_M(t), \psi_{n,m}(x) \rangle - a_{n,m} \\
&= a_{n,m} - a_{n,m} = 0,
\end{aligned}$$

hence $g(t) = f(t)$ and the proof is complete.

According to **Theorem 1**, the infinite series in Eq. (2) can be approximated by

$$f(t) \cong f_M(t) = \sum_{n=0}^{2^l-1} \sum_{m=0}^{M-1} a_{n,m} \psi_{n,m}(t) = A^{\mathrm{T}} \Psi(t), \tag{4}$$

where

$$a_{n,m} = \int_0^T f(t) \psi_{n,m}(t) dt, \;\; n = 0,1,...,2^l - 1, \;\; m = 0,1,...,M-1,$$

and

$$A = [a_{0,0}, a_{0,1}, ..., a_{0,M-1}, a_{1,0}, ..., a_{1,M-1}, ..., a_{(2^l-1),0}, ..., a_{(2^l-1),M-1}]^{\mathrm{T}},$$

$$\Psi = [\psi_{0,0}, \psi_{0,1}, ..., \psi_{0,M-1}, \psi_{1,0}, ..., \psi_{1,M-1}, ..., \psi_{(2^l-1),0}, ..., \psi_{(2^l-1),M-1}]^{\mathrm{T}}.$$

The following lemma gives an upper bound for the error of the estimate.

Lemma 1. Suppose that $f : [0,T] \to \mathbb{R}$ is M times continuously differentible and let $A^{\mathrm{T}}\Psi$ approximates f. Then an upper bound for the error of approximation is as follows:

$$\|f - A^{\mathrm{T}}\Psi\|_2 \leq \frac{ST^{\frac{2M+1}{2}}}{M!\sqrt{2M+1}(2^l)^{\frac{2M+1}{2}}}, \tag{5}$$

where

$$S = \max_{t\in[0,T]} |f^{(M)}(t)|.$$

Proof. A Taylor polynomial approximation for $f(t)$ is

$$f(t) = \underbrace{f(a) + f'(a)(t-a) + ... + f^{(M-1)}(a)\frac{(t-a)^{(M-1)}}{(M-1)!}}_{I(t)} + f^{(M)}(\eta)\frac{(t-a)^M}{(M)!},$$

where $a = \frac{nT}{2^l}$ and $\eta \in (0,T)$. We know that

$$|f(t) - I(t)| \leq \left|f^{(M)}(\eta)\right| \frac{(t-\frac{nT}{2^l})^M}{(M)!},\ \eta \in (0,T). \tag{6}$$

Since $A^{\mathrm{T}}\Psi$ is a polynomial of degree $M-1$ that approximates f with the minimum mean error bound, we have by Eq. (6)

$$\begin{aligned}
\|f - A^{\mathrm{T}}\Psi\|_2^2 \leq \|f - I\|_2^2 &= \int_0^T |f(t) - I(t)|^2 dt \\
&\leq \int_0^T [f^{(M)}(\eta)\frac{(t-\frac{nT}{2^l})^M}{(M)!}]^2 dx \\
&= \sum_{n=0}^{2^l-1} \int_{\frac{nT}{2^l}}^{\frac{(n+1)T}{2^l}} [f^{(M)}(\eta)\frac{(t-\frac{nT}{2^l})^M}{(M)!}]^2 dt \\
&\leq \frac{S^2}{(M!)^2} \sum_{n=0}^{2^l-1} \int_{\frac{nT}{2^l}}^{\frac{(n+1)T}{2^l}} (t-\frac{nT}{2^l})^{2M} dt \\
&= \frac{S^2 T^{2M+1}}{(M!)^2 (2^l)^{2M+1}(2M+1)}.
\end{aligned}$$

Taking squre roots, we have Eq. (5).

The upper bound of the error depends on

$$\frac{T^{\frac{2M+1}{2}}}{M!\sqrt{2M+1}(2^l)^{\frac{2M+1}{2}}},$$

which shows that as M increases, the error approaches zero rapidly. This is an advantages of Legendre multiwavelet function approximations.

Method of solution

First, we rewrite the system (1) as follows:

$$\begin{cases} R_1(T,I,V) = D^{\alpha_1}(T) - q + \alpha T - rT\left(1 - \frac{T+I}{T_{max}}\right) + kVT, \\ R_2(T,I,V) = D^{\alpha_2}(I) - kVT + \beta I, \\ R_3(T,I,V) = D^{\alpha_3}(V) - \mu\beta I + \gamma V. \end{cases} \tag{7}$$

By Eq. (4) we can approximate T, I and V by using the truncated series as

$$T_M(t) = \sum_{n=0}^{2^l-1}\sum_{m=0}^{M-1} a_{n,m}\psi_{n,m}(t),$$

$$I_N(t) = \sum_{n=0}^{2^l-1}\sum_{m=0}^{N-1} b_{n,m}\psi_{n,m}(t),$$

$$V_K(t) = \sum_{n=0}^{2^l-1}\sum_{m=0}^{l-1} c_{n,m}\psi_{n,m}(t).$$

To employ the Galerkin method, we put

$$\begin{aligned} &\langle R_1(T_M,I_N,V_K),\psi_{n,m}\rangle = 0, \ \ n = 0,1,...,2^l-1, \ \ m = 0,1,...,M-2, \\ &\langle R_2(T_M,I_N,V_K),\psi_{n,m}\rangle = 0, \ \ n = 0,1,...,2^l-1, \ \ m = 0,1,...,N-2, \\ &\langle R_3(T_M,I_N,V_K),\psi_{n,m}\rangle = 0, \ \ n = 0,1,...,2^l-1, \ \ m = 0,1,...,K-2, \end{aligned} \tag{8}$$

and

$$T_M(0) = T_0,$$

$$I_N(0) = I_0,$$

$$V_K(0) = V_0.$$

Now we have a nonlinear algebraic system of equations with unknown coefficients and we may approximate the solutions of the system (1) applying Newton's iterative method to Eqs. (8).

We can check the accuracy of these solutions by substituting $T_M(t)$, $I_N(t)$ and $V_K(t)$ in Eq. (7). Hence, given for large M, N and K, at any of $t = t_s \in [0,T]$, $s \in \mathbb{N}$, we should have:

$$|R_i(T_M,I_N,V_K)| \cong 0$$

or

$$|R_i(T_M,I_N,V_K)| \leq 10^{-l_s}, \ i = 1,2,3$$

(l_s is a positive integer).

If $\max 10^{-l_s} = 10^{-l}$ (l is a positive integer) is considered, then the elements M, N and K are increased until the values of $|R_i(T_M, I_N, V_K)|$, $i = 1,2,3$, at each of t_s, $s \in \mathbb{N}$ become smaller than the considered 10^{-l}.

Numerical results

We consider the fractional model of HIV with:

$q = 0.1$, $\alpha = 0.02$, $r = 3$, $T_{max} = 1500$, $k = 0.0027$, $\beta = 0.3$, $\mu = 10$, $\gamma = 2.4$,
$T_0 = V_0 = 0.1$, $I_0 = 0$ and set $l = 0$.

Fig. 1 displays the comparison of $T(t)$, $I(t)$ and $V(t)$ for different values of α_i, $i = 1,2,3$.

The comparison of errors, $|R_i|$ for $\alpha_i = 1$, $i = 1,2,3$, and different values of M,N and K are shown in Fig. 2.

In the case when $M = N = K = 12$, the approximate solutions $T_M(t)$, $I_N(t)$ as well as corresponding errors $|R_1|$, $|R_2|$ and $|R_3|$ are given in Tables 1-3, for the three values of α_i, $i = 1,2,3$, respectively.

Table 1. Numerical results by the present method for $\alpha_i = 1$, $i = 1,2,3$

t	$T_{12}(t)$	$I_{12}(t)$	$V_{12}(t)$	$\|R_1\|$	$\|R_2\|$	$\|R_3\|$
0.0	0.1	0	0.1	0	0	0
0.2	0.2088080843	0.603270224e-5	0.061879843224	9.9e-11	8.9e-16	1.0e-13
0.4	0.4062405428	0.131583409e-4	0.038294887773	3.1e-10	5.1e-16	3.9e-13
0.6	0.7644238985	0.212237854e-4	0.023704550045	2.9e-10	7.7e-17	3.9e-13
0.8	1.4140468519	0.301774201e-4	0.014680363684	5.9e-11	2.7e-16	1.0e-13
1.0	2.5915948517	0.400378155e-4	0.009100844997	1.8e-9	1.9e-15	2.1e-12

Table 2. Numerical results by the present method for $\alpha_i = 0.99$, $i = 1,2,3$, and $M = N = K = 12$

t	$T_{12}(t)$	$I_{12}(t)$	$V_{12}(t)$	$\|R_1\|$	$\|R_2\|$	$\|R_3\|$
0.0	0.1	0	0.1	0	0	0
0.2	0.2118151955	0.61698792516e-5	0.061266390955	1.3e-4	7.7e-9	8.1e-5
0.4	0.4134375802	0.13366187112e-4	0.037794686364	9.6e-6	2.6e-9	1.6e-5
0.6	0.7781041191	0.21467146845e-4	0.023382395777	9.2e-5	6.2e-9	6.7e-5
0.8	1.4370229054	0.30426220288e-4	0.014497426535	5.4e-5	1.3e-9	3.6e-5
1.0	2.6259351154	0.40263784819e-4	0.009006443439	2.9e-4	4.6e-8	2.3e-4

Table 3. Numerical results by the present method for $\alpha_i = 0.9$, $i = 1,2,3$, and $M = N = K = 12$

t	$T_{12}(t)$	$I_{12}(t)$	$V_{12}(t)$	$\|R_1\|$	$\|R_2\|$	$\|R_3\|$
0.0	0.1	0	0.1	0	0	0
0.2	0.2462987100	0.76613468661e-5	0.055079132183	8.9e-4	7.7e-8	1.5e-3
0.4	0.4975077831	0.15615612059e-4	0.032860997206	2.8e-4	4.4e-8	2.2e-4
0.6	0.9416409759	0.24141356480e-4	0.020172877424	8.7e-4	7.6e-8	1.1e-3
0.8	1.7209440187	0.33258878322e-4	0.012601296083	4.5e-4	6.5e-9	6.5e-4
1.0	3.0759133504	0.43018985522e-4	0.007945335547	3.1e-3	6.5e-7	3.7e-3

Tables 4-6 display a comparison of the error of our method (for $M = N = K = 12$) with the LADM [12], the Runge-Kutta method, the variational iteration method (VIM) [9] and the Bessel collocation method [19].

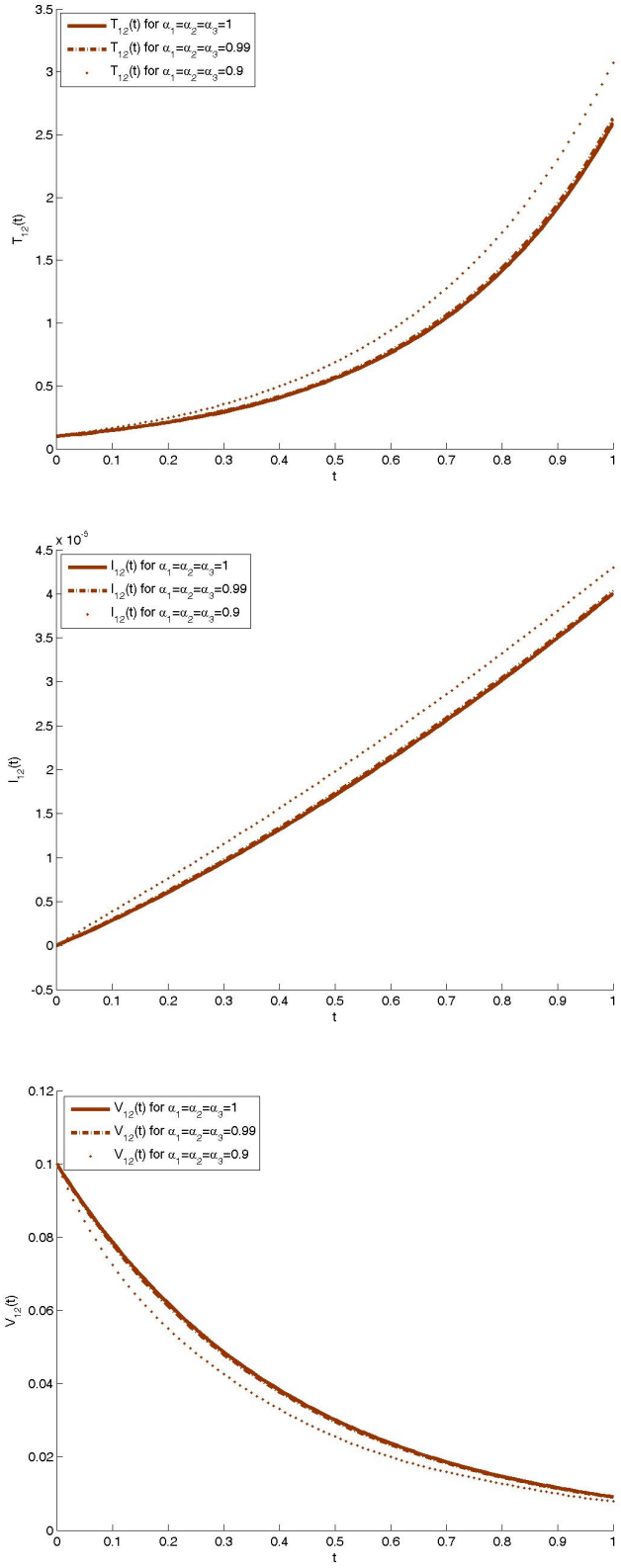

Fig. 1 Top: The comparison of $T_M(t)$ for different values of α_i, $i = 1,2,3$, and $M = 12$, Middle: The comparison of $I_N(t)$ for different values of α_i, $i = 1,2,3$, and $N = 12$, Bottom: The comparison of $V_K(t)$ for different values of α_i, $i = 1,2,3$, and $K = 12$.

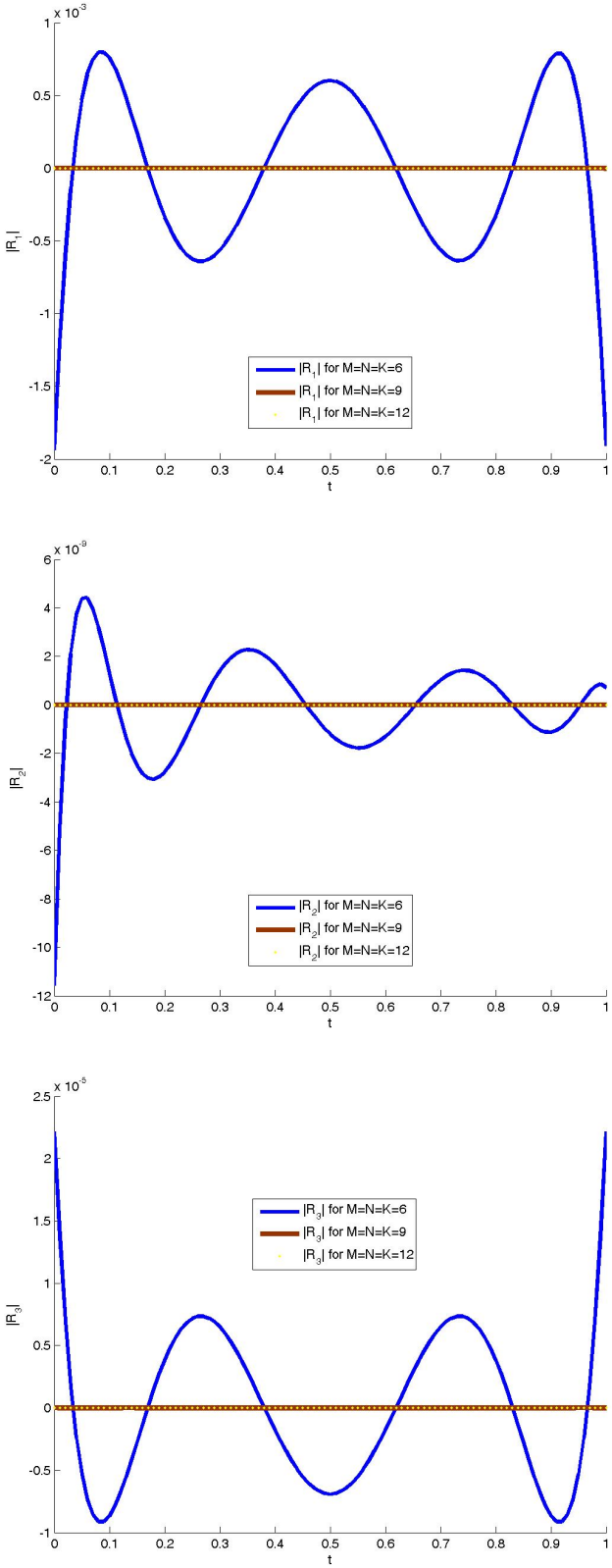

Fig. 2 Top: The comparison of $|R_1|$ for different values of M,N,K and $\alpha_i = 1$, $i = 1,2,3$, Middle:The comparison of $|R_2|$ for different values of M,N,K and $\alpha_i = 1$, $i = 1,2,3$, Bottom: The comparison of $|R_3|$ for different values of M,N,K and $\alpha_i = 1$, $i = 1,2,3$.

Table 4. Numerical comparison for $T(t)$

t	LADM [12]	Runge-Kutta	VIM [9]	Method in [19]	Present method
0.0	0.1	0.1	0.1	0.1	0.1
0.2	0.2088072731	0.2088080833	0.2088073214	0.2038616561	0.2088080843
0.4	0.4061052625	0.4062405393	0.4061346587	0.3803309335	0.4062405428
0.6	0.7611467713	0.7644238890	0.7624530350	0.6954623767	0.7644238985
0.8	1.3773198590	1.4140468310	1.3978805880	1.2759624442	1.4140468519
1.0	2.3291697610	2.5915948020	2.5067466690	2.3832277428	2.5915948517

Table 5. Numerical comparison for $I(t)$

t	LADM [12]	Runge-Kutta	VIM [9]	Method in [19]	Present method
0.0	0	0	0	0	0
0.2	0.603270728e-5	0.603270215e-5	0.6032634366e-5	0.624787210e-5	0.603270224e-5
0.4	0.131591617e-4	0.131583407e-4	0.1314878543e-4	0.129355222e-4	0.131583409e-4
0.6	0.212683688e-4	0.212237850e-4	0.2101417193e-4	0.203526718e-4	0.212237854e-4
0.8	0.300691867e-4	0.301774195e-4	0.2795130456e-4	0.283730212e-4	0.301774201e-4
1.0	0.398736542e-4	0.400378146e-4	0.2431562317e-4	0.369084236e-4	0.400378155e-4

Table 6. Numerical comparison for $V(t)$

t	LADM [12]	Runge-Kutta	VIM [9]	Method in [19]	Present method
0.0	0.1	0.1	0.1	0.1	0.1
0.2	0.06187996025	0.06187984331	0.06187995314	0.06187991856	0.061879843224
0.4	0.03831324883	0.03829488788	0.03830820126	0.03829493490	0.038294887773
0.6	0.02439174349	0.02370455014	0.02392029257	0.02370431860	0.023704550045
0.8	0.009967218934	0.01468036377	0.01621704553	0.01467956982	0.014680363684
1.0	0.003305076447	0.009100845043	0.01608418711	0.02370431861	0.009100844997

Conclusion

In this paper, we have presented a numerical method to solve a fractional model for HIV infection of $CD4^+T$ cells. We successfully applied the Galerkin method based on Legendre multiwavelet functions and obtained very good approximate solutions using only a few terms. The main advantage of the proposed algorithm is that by with only a small number of adding terms of the Legendre multiwavelet functions, we get much better approximations to unknown functions. Comparisons between our approximations with approximate solutions achieved by other methods were carried out to confirm the validity and applicability of the new algorithm.

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