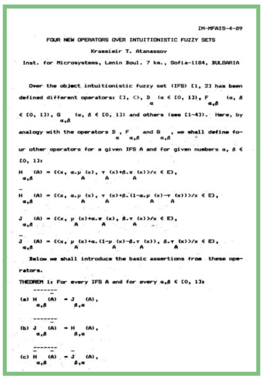
Four New Operators over Intuitionistic Fuzzy Sets

Krassimir T. Atanassov*

How to cite:

Atanassov K. T. Four New Operators over Intuitionistic Fuzzy Sets, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1989, Preprint IM-MFAIS-4-89. Reprinted: Int J Bioautomation, 2016, 20(S1), S55-S62.



Over the object intuitionistic fuzzy set (IFS) [1, 2] has been defined different operators: □, ◇, D_α ($\alpha \in$ [0, 1]), $F_{\alpha,\beta}$ ($\alpha, \beta \in$ [0, 1]), $G_{\alpha,\beta}$ ($\alpha, \beta \in$ [0, 1]), and others (see [1-4]). Here, by analogy with the operators D_α, $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$, we shall define four other operators for a given IFS A and for given numbers $\alpha, \beta \in$ [0, 1]:

$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle \mid x \in E\},$$

$$\bar{H}_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle \mid x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle \mid x \in E\},$$

$$\bar{J}_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle \mid x \in E\}.$$

Below we shall introduce the basic assertions from these operators.

Theorem 1: For every IFS A and for every $\alpha, \beta \in$ [0, 1]:

(a) $\overline{H_{\alpha,\beta}(\bar{A})} = J_{\beta,\alpha}(A)$,

(b) $\overline{J_{\alpha,\beta}(\bar{A})} = H_{\beta,\alpha}(A)$,

(c) $\overline{\bar{H}_{\alpha,\beta}(\bar{A})} = \bar{J}_{\beta,\alpha}(A)$,

(d) $\overline{\bar{J}_{\alpha,\beta}(\bar{A})} = \bar{H}_{\beta,\alpha}(A)$.

Proof: (a)
$$\overline{H_{\alpha,\beta}(\bar{A})} = \overline{\{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle \mid x \in E\}} = \{\langle x, \mu_A(x) + \beta.\pi_A(x), \alpha.\nu_A(x)\rangle \mid x \in E\} = J_{\beta,\alpha}(A).$$

(b)-(d) are proved analogically. □

---

* Current affiliation: Bioinformatics and Mathematical Modelling Department
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria, E-mail: krat@bas.bg

Theorem 2: For every IFS A and for every $\alpha, \beta \in [0,1]$:

(a) $H_{\alpha,\beta}(A) = F_{0,\beta}(A) \cap G_{\alpha,1}(A)$,

(b) $J_{\alpha,\beta}(A) = F_{\beta,0}(A) \cup G_{1,\alpha}(A)$,

(c) $\overline{H}_{\alpha,\beta}(A) = F_{0,\beta}(G_{\alpha,1}(A))$,

(d) $\overline{H}_{\alpha,\beta}(A) = F_{0,\beta}(G_{\alpha,1}(A))$.

Proof:

(a) $H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}$
$= \{\langle x, \min(\mu_A(x), \alpha.\mu_A(x)), \max(\nu_A(x) + \beta.\pi_A(x), \nu_A(x))\rangle | x \in E\} = F_{0,\beta}(A) \cap G_{\alpha,1}(A)$.

(b) is proved analogically.

(c) $\overline{H}_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$
$= F_{0,\beta}(\{\langle x, \alpha.\mu_A(x), \nu_A(x))\rangle | x \in E\}) = F_{0,\beta}(G_{\alpha,1})$.

(d) is proved analogically. □

Theorem 3: For every IFSs A and B and for every $\alpha, \beta \in [0,1]$:

(a) $H_{\alpha,\beta}(A \cap B) \subset H_{\alpha,\beta}(A) \cap H_{\alpha,\beta}(B)$,

(b) $H_{\alpha,\beta}(A \cup B) \supset H_{\alpha,\beta}(A) \cup H_{\alpha,\beta}(B)$,

(c) $H_{\alpha,\beta}(A + B) \subset H_{\alpha,\beta}(A) + H_{\alpha,\beta}(B)$,

(d) $H_{\alpha,\beta}(A.B) \supset H_{\alpha,\beta}(A).H_{\alpha,\beta}(B)$,

(e) $J_{\alpha,\beta}(A \cap B) \supset J_{\alpha,\beta}(A) \cap J_{\alpha,\beta}(B)$,

(f) $J_{\alpha,\beta}(A \cup B) \subset J_{\alpha,\beta}(A) \cup J_{\alpha,\beta}(B)$,

(g) $J_{\alpha,\beta}(A + B) \supset J_{\alpha,\beta}(A) + J_{\alpha,\beta}(B)$,

(h) $J_{\alpha,\beta}(A.B) \subset J_{\alpha,\beta}(A).J_{\alpha,\beta}(B)$,

(i) $\overline{H}_{\alpha,\beta}(A \cap B) \subset \overline{H}_{\alpha,\beta}(A) \cap \overline{H}_{\alpha,\beta}(B)$,

(j) $\overline{H}_{\alpha,\beta}(A \cup B) \supset \overline{H}_{\alpha,\beta}(A) \cup \overline{H}_{\alpha,\beta}(B)$,

(k) $\overline{H}_{\alpha,\beta}(A + B) \subset \overline{H}_{\alpha,\beta}(A) + \overline{H}_{\alpha,\beta}(B)$,

(l) $\overline{H}_{\alpha,\beta}(A.B) \supset \overline{H}_{\alpha,\beta}(A).\overline{H}_{\alpha,\beta}(B)$,

(m) $\overline{J}_{\alpha,\beta}(A \cap B) \supset \overline{J}_{\alpha,\beta}(A) \cap \overline{J}_{\alpha,\beta}(B)$,

(n) $\overline{J}_{\alpha,\beta}(A \cup B) \subset \overline{J}_{\alpha,\beta}(A) \cup \overline{J}_{\alpha,\beta}(B)$,

(o) $\overline{J}_{\alpha,\beta}(A+B) \supset \overline{J}_{\alpha,\beta}(A) + \overline{J}_{\alpha,\beta}(B)$,

(p) $\overline{J}_{\alpha,\beta}(A.B) \subset \overline{J}_{\alpha,\beta}(A).\overline{J}_{\alpha,\beta}(B)$.

Proof:

(a) $H_{\alpha,\beta}(A \cap B) = H_{\alpha,\beta}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\})$
$= \{\langle x, \alpha.\min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) + \beta.(1 - \min(\mu_A(x), \mu_B(x)) - \max(\nu_A(x), \nu_B(x)))\rangle | x \in E\}$
$H_{\alpha,\beta}(A) \cap H_{\alpha,\beta}(B) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \cap \{\langle x, \alpha.\mu_B(x), \nu_B(x) + \beta.\pi_B(x)\rangle | x \in E\}$
$= \{\langle x, \min(\alpha.\mu_A(x), \alpha.\mu_B(x)), \max(\nu_A(x) + \beta.(1 - \min(\mu_A(x) - \nu_A(x), \nu_B(x) + \beta.(1 - \min(\mu_B(x) - \nu_B(x)))\rangle | x \in E\}$

The second terms (the degrees of membership) in both sets are equal and for the third terms is valid the following inequality:

$\max(\nu_A(x), \nu_B(x)) + \beta.(1 - \min(\mu_A(x), \mu_B(x)) - \max(\nu_A(x), \nu_B(x)))$
$- \max(\nu_A(x) + \beta.(1 - \min(\mu_A(x) - \nu_A(x)), \nu_B(x) + \beta.(1 - \min(\mu_B(x) - \nu_B(x)))$
$= (1-\beta).\max(\nu_A(x), \nu_B(x)) + \beta.\max(1 - (\mu_A(x), 1 - \mu_B(x)) - \max((1-\beta).\nu_A(x) + \beta.(1 - \min(\mu_A(x)), (1-\beta).\nu_B(x) + \beta.(1 - \min(\mu_B(x)))) \geq 0$,

because for every real numbers a, b, c and d:

$\max(a,b) + \max(c,d) - \max(a+c, b+d) \geq 0$.

(k) $\overline{H}_{\alpha,\beta}(A+B)$
$= \overline{H}_{\alpha,\beta}(\{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x) | x \in E\})$
$= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x) + \beta.(1 - \alpha.(\mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)) - \nu_A(x)\nu_B(x))\rangle | x \in E\}$.

$H_{\alpha,\beta}(A) + H_{\alpha,\beta}(B)$
$= \{\langle x, \mu_A(x), \nu_A(x) + \beta.\pi_A(x) | x \in E\} + \{\langle x, \mu_B(x), \nu_B(x) + \beta.\pi_B(x) | x \in E\}$
$= \{\langle x, \alpha.\mu_A(x) + \alpha.\mu_B(x) - \alpha^2.\mu_A(x).\mu_B(x), (\nu_A(x) + \beta.\pi_A(x)).(\nu_B(x) + \beta.\pi_B(x)) | x \in E\}$.

Obviously the second term of the first set is greater than the second term of the second set. On the other hand:

$(\nu_A(x) + \beta.\pi_A(x)).(\nu_B(x) + \beta.\pi_B(x)) - \nu_A(x).\nu_B(x) + \beta.(1 - \alpha.(\mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)) - \nu_A(x)\nu_B(x)))$
$\geq \beta.(1 - \alpha.(\mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x) - \nu_A(x).\nu_B(x)))$
$\geq \beta.(1 - \mu_A(x) - \mu_B(x) + \mu_A(x).\mu_B(x) - \nu_A(x).\nu_B(x)))$
≥ 0.

The other assertions are proved analogically. □

Theorem 4: For every IFS A and for every $\alpha, \beta \in [0, 1]$:

(a) $\Box H_{\alpha,\beta}(A) \subset H_{\alpha,\beta}(\Box A)$,

(b) $\Diamond H_{\alpha,\beta}(A) \subset_{\Diamond} H_{\alpha,\beta}(\Diamond A)$,

(c) $J_{\alpha,\beta}(\Box A) \subset_{\Box} \Box J_{\alpha,\beta}(A)$,

(d) $J_{\alpha,\beta}(\Diamond A) \subset \Diamond J_{\alpha,\beta}(A)$,

(e) $\Box \overline{H}_{\alpha,\beta}(A) \subset \overline{H}_{\alpha,\beta}(\Box A)$,

(f) $\Diamond \overline{H}_{\alpha,\beta}(A) \subset_{\Diamond} \overline{H}_{\alpha,\beta}(\Diamond A)$,

(g) $\overline{J}_{\alpha,\beta}(\Box A) \subset_{\Box} \Box \overline{J}_{\alpha,\beta}(A)$,

(h) $\overline{J}_{\alpha,\beta}(\Diamond A) \subset \Diamond \overline{J}_{\alpha,\beta}(A)$.

Proof:

(h) $\overline{J}_{\alpha,\beta}(\Diamond A) = \{\langle x, 1-\beta.\nu_A(x), \beta.\nu_A(x)\rangle | x \in E\}$ and
$\overline{J}_{\alpha,\beta}(A) = \{\langle x, 1-\nu_A(x)+\alpha.(\nu_A(x)-\beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\}$.

From the equality of the third components of the two IFSs and from

$$1-\beta.\nu_A(x)-(1-\nu_A(x)+\alpha.(\nu_A(x)-\beta.\nu_A(x)) = \nu_A(x).(1-\alpha).(1-\beta) \geq 0$$

follows the validity of (h).

The other assertions are proved analogically. □

Theorem 5: For every IFS A and for every $\alpha, \beta \in [0, 1]$ and for every $\gamma, \delta \in [0, 1]$ for which $0 \leq \gamma+\delta \leq 1$:

(a) $F_{\gamma,\delta}(H_{\alpha,\beta}(A)) \subset_{\Diamond} H_{\alpha,\beta}(F_{\gamma,\delta}(A))$,

(b) $J_{\alpha,\beta}(F_{\gamma,\delta}(A)) \subset_{[]} F_{\gamma,\delta}(J_{\alpha,\beta}(A))$.

Proof:

(a) $F_{\gamma,\delta}(H_{\alpha,\beta}(A))$
$= \{\langle x, \alpha.\mu_A(x)+\gamma.(1-\alpha.\mu_A(x)-\nu_A(x)-\beta.\pi_A(x), \nu_A(x)+\beta.\pi_A(x)+\delta.(1-\alpha.\mu_A(x)$
$-\nu_A(x)-\beta.\pi_A(x))\rangle | x \in E\}$,
$H_{\alpha,\beta}(F_{\gamma,\delta}(A))$
$= \{\langle x, \alpha.(\mu_A(x)+\gamma.\pi_A(x)), \nu_A(x)+\delta.\pi_A(x)+\beta.(1-\mu_A(x)-\gamma.\pi_A(x)-\nu_A(x)$
$-\delta.\pi_A(x))\rangle | x \in E\}$.

From:

$\beta.\pi_A(x) + \delta - \alpha.\delta.\mu_A(x) - \delta.\nu_A(x) - \beta - \delta.\pi_A(x) + \beta.\mu_A(x) + \beta.\gamma.\pi_A(x) + \beta.\nu_A(x)$
$= \delta.(1 - \alpha.\mu_A(x) - \nu_A(x) - \pi_A(x)) + \beta.\gamma.\pi_A(x)$
$\geq \delta.(1 - \mu_A(x) - \nu_A(x) - \pi_A(x)) + \beta.\gamma.\pi_A(x)$
$\geq \beta.\gamma.\pi_A(x) \geq 0$
follows the validity of (a).

(b) is proved analogically. □

Other relations between the operators $F_{\gamma,\delta}$ (resp. D_α) and $H_{\alpha,\beta}$ and $J_{\alpha,\beta}$ are not valid.

Theorem 6: For every IFS A and for every $\alpha, \beta, \gamma, \delta \in [0,1]$:

(a) $H_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(H_{\alpha,\beta}(A))$,

(b) $J_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(J_{\alpha,\beta}(A))$,

(c) $\overline{H}_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(\overline{H}_{\alpha,\beta}(A))$,

(d) $\overline{J}_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(\overline{J}_{\alpha,\beta}(A))$.

Proof:

(a) $H_{\alpha,\beta}(G_{\gamma,\delta}(A))$
$= \{\langle x, \alpha.\gamma.\mu_A(x), \delta.\nu_A(x) + \beta.(1 - \gamma.\mu_A(x) - \delta.\nu_A(x))\rangle | x \in E\}$,

$(G_{\gamma,\delta}(H_{\alpha,\beta}(A))$
$= \{\langle x, \alpha.\gamma.\mu_A(x), \delta.\nu_A(x) + \beta.\delta.(1 - \mu_A(x) - \nu_A(x))\rangle | x \in E\}$.

The second terms in both sets are equal and for the third terms is valid the following inequality:
$\delta.\nu_A(x) + \beta.(1 - \gamma.\mu_A(x) - \delta.\nu_A(x)) - (\delta.\nu_A(x) + \beta.\delta.(1 - \mu_A(x) - \nu_A(x)))$
$= \beta.(1 - \gamma.\mu_A(x) - \delta.\nu_A(x) - \delta.(1 - \mu_A(x) - \nu_A(x))$
$= \beta.(1 - \gamma.\mu_A(x) - \delta + \delta.\mu_A(x))$

if $\gamma \geq \delta$:
$\geq \beta.(\gamma - \delta).(1 - \mu_A(x)) \geq 0$;

if $\gamma < \delta$:
$\geq \beta.(1 - \delta + \mu_A(x).(\delta - \gamma)) \geq 0$;

i.e., (a) is valid.

The other assertions are proved analogically. □

References

1. Atanassov K. (1986). Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 20, 87-96.
2. Atanassov K. (1988, 2016). Review and New Results on Intuitionistic Fuzzy Sets, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1988, Preprint IM-MFAIS-1-88, Reprinted: Int J Bioautomation, 20(S1), S7-S16.
3. Atanassov K. (1989). More on Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 33(1), 37-45.
4. Atanassov K. (1988). Two Operators on Intuitionistic Fuzzy Sets, Comptes Rendus de l'Academie Bulgare des Sciences, 41(5), 35-38.

Original references as presented in Preprint IM-MFAIS-4-89

2. Atanassov K. Review and new results on intuitionistic fuzzy sets. Preprint IM-MFAIS-1-88, Sofia, 1988
3. Atanassov K. More on intuitionistic fuzzy sets. Fuzzy sets and Systems (in press).

Facsimiles

IM-MFAIS-4-89

FOUR NEW OPERATORS OVER INTUITIONISTIC FUZZY SETS

Krassimir T. Atanassov

Inst. for Microsystems, Lenin Boul. 7 km., Sofia-1184, BULGARIA

Over the object intuitionistic fuzzy set (IFS) [1, 2] has been defined different operators: [], <>, D_α ($\alpha \in [0, 1]$), $F_{\alpha,\beta}$ ($\alpha, \beta \in [0, 1]$), $G_{\alpha,\beta}$ ($\alpha, \beta \in [0, 1]$) and others (see [1-4]). Here, by analogy with the operators D_α, $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$, we shall define four other operators for a given IFS A and for given numbers $\alpha, \beta \in [0, 1]$:

$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \tau_A(x)+\beta.\pi_A(x)\rangle / x \in E\},$$

$$\bar{H}_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \tau_A(x)+\beta.(1-\alpha.\mu_A(x)-\tau_A(x))\rangle / x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x)+\alpha.\pi_A(x), \beta.\tau_A(x)\rangle / x \in E\},$$

$$\bar{J}_{\alpha,\beta}(A) = \{\langle x, \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\tau_A(x)), \beta.\tau_A(x)\rangle / x \in E\},$$

Below we shall introduce the basic assertions from these operators.

THEOREM 1: For every IFS A and for every $\alpha,\beta \in [0, 1]$:

(a) $\overline{H_{\alpha,\beta}(\bar{A})} = J_{\beta,\alpha}(A)$,

(b) $\overline{J_{\alpha,\beta}(\bar{A})} = H_{\beta,\alpha}(A)$,

(c) $\overline{\bar{H}_{\alpha,\beta}(\bar{A})} = \bar{J}_{\beta,\alpha}(A)$,

Page 1

- 2 -

(d) $\overline{\bar{J}_{\alpha,\beta}(\bar{A})} = \bar{H}_{\beta,\alpha}(A)$.

Proof: (a) $\overline{H_{\alpha,\beta}(\bar{A})} = \overline{\{\langle x, \alpha.\tau_A(x), \mu_A(x)+\beta.\pi_A(x)\rangle / x \in E\}}$

$= \{\langle x, \mu_A(x)+\beta.\pi_A(x), \alpha.\pi_A(x)\rangle / x \in E\} = J_{\beta,\alpha}(x)$.

(b) - (d) are proved analogically.

THEOREM 2: For every IFS A and for every $\alpha,\beta \in [0, 1]$:

(a) $H_{\alpha,\beta}(A) = F_{0,\beta}(A) \cap G_{\alpha,1}(A)$,

(b) $J_{\alpha,\beta}(A) = F_{\beta,0}(A) \cup G_{1,\alpha}(A)$,

(c) $\bar{H}_{\alpha,\beta}(A) = F_{0,\beta}(G_{\alpha,1}(A))$,

(d) $\bar{J}_{\alpha,\beta}(A) = F_{\beta,0}(G_{1,\alpha}(A))$.

Proof: (a) $H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \tau_A(x)+\beta.\pi_A(x)\rangle / x \in E\}$

$= \{\langle x, \min(\mu_A(x), \alpha.\mu_A(x)), \max(\tau_A(x)+\beta.\pi_A(x), \tau_A(x))\rangle / x \in E\}$

$= F_{0,\beta}(A) \cap G_{\alpha,1}(A)$.

(b) is proved analogically.

(c) $\bar{H}_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \tau_A(x)+\beta.(1-\alpha.\mu_A(x)-\tau_A(x))\rangle / x \in E\}$

$= F_{0,\beta}(\{\langle x, \alpha.\mu_A(x), \tau_A(x)\rangle / x \in E\})$

$= F_{0,\beta}(G_{\alpha,1}(A))$.

(d) is proved analogically.

THEOREM 3: For every two IFSs A and B and for every $\alpha,\beta \in [0, 1]$:

(a) $H_{\alpha,\beta}(A \cap B) \subset H_{\alpha,\beta}(A) \cap H_{\alpha,\beta}(B)$,

Page 2

- 3 -

(b) $H_{\alpha,\beta}(A \cup B) \supset H_{\alpha,\beta}(A) \cup H_{\alpha,\beta}(B)$,

(c) $H_{\alpha,\beta}(A + B) \subset H_{\alpha,\beta}(A) + H_{\alpha,\beta}(B)$,

(d) $H_{\alpha,\beta}(A . B) \supset H_{\alpha,\beta}(A) . H_{\alpha,\beta}(B)$,

(e) $J_{\alpha,\beta}(A \cap B) \supset J_{\alpha,\beta}(A) \cap J_{\alpha,\beta}(B)$,

(f) $J_{\alpha,\beta}(A \cup B) \subset J_{\alpha,\beta}(A) \cup J_{\alpha,\beta}(B)$,

(g) $J_{\alpha,\beta}(A + B) \supset J_{\alpha,\beta}(A) + J_{\alpha,\beta}(B)$,

(h) $J_{\alpha,\beta}(A . B) \subset J_{\alpha,\beta}(A) . J_{\alpha,\beta}(B)$,

(i) $\bar{H}_{\alpha,\beta}(A \cap B) \subset \bar{H}_{\alpha,\beta}(A) \cap \bar{H}_{\alpha,\beta}(B)$,

(j) $\bar{H}_{\alpha,\beta}(A \cup B) \supset \bar{H}_{\alpha,\beta}(A) \cup \bar{H}_{\alpha,\beta}(B)$,

(k) $\bar{H}_{\alpha,\beta}(A + B) \subset \bar{H}_{\alpha,\beta}(A) + \bar{H}_{\alpha,\beta}(B)$,

(l) $\bar{H}_{\alpha,\beta}(A . B) \supset \bar{H}_{\alpha,\beta}(A) . \bar{H}_{\alpha,\beta}(B)$,

(m) $\bar{J}_{\alpha,\beta}(A \cap B) \supset \bar{J}_{\alpha,\beta}(A) \cap \bar{J}_{\alpha,\beta}(B)$,

(n) $\bar{J}_{\alpha,\beta}(A \cup B) \subset \bar{J}_{\alpha,\beta}(A) \cup \bar{J}_{\alpha,\beta}(B)$,

(o) $\bar{J}_{\alpha,\beta}(A + B) \supset \bar{J}_{\alpha,\beta}(A) + \bar{J}_{\alpha,\beta}(B)$,

(p) $\bar{J}_{\alpha,\beta}(A . B) \subset \bar{J}_{\alpha,\beta}(A) . \bar{J}_{\alpha,\beta}(B)$.

Proof: (a) $H_{\alpha,\beta}(A \cap B)$

$= H_{\alpha,\beta}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\tau_A(x), \tau_B(x))\rangle / x \in E\})$

Page 3

- 4 -

$= \{\langle x, \alpha.\min(\mu_A(x), \mu_B(x)), \max(\tau_A(x), \tau_B(x))+\beta.(1-\min(\mu_A(x),$

$\mu_B(x))-\max(\tau_A(x), \tau_B(x))\rangle / x \in E\}$

$H_{\alpha,\beta}(A) \cap H_{\alpha,\beta}(B)$

$= \{\langle x,\alpha.\mu_A(x), \tau_A(x)+\beta.\pi_A(x)\rangle x \in E\} \cap \{\langle x,\alpha.\mu_A(x), \tau_A(x)+$

$\beta.\pi_A(x)\rangle x \in E\}$

$= \{\langle x, \min(\alpha.\mu_A(x), \alpha.\mu_B(x)), \max(\tau_A(x)+\beta.(1-\min(\mu_A(x)-\tau_A(x),$

$\tau_B(x)+\beta.(1-\min(\mu_B(x)-\tau_B(x))\rangle / x \in E\}$

The second terms (the degrees of membership) in both sets are equal and for the third terms is valid the following inequality:

$\max(\tau_A(x), \tau_B(x))+\beta.(1-\min(\mu_A(x), \mu_B(x))-\max(\tau_A(x), \tau_B(x)))$

$- \max(\tau_A(x)+\beta.(1-\min(\mu_A(x)-\tau_A(x)), \tau_B(x)+\beta.(1-\min(\mu_B(x)-\tau_B(x)))$

$= (1-\beta).\max(\tau_A(x), \tau_B(x))+\beta.\max(1-(\mu_A(x), 1-\mu_B(x))-\max((1-\beta).$

$\tau_A(x)+\beta.(1-\min(\mu_A(x)), (1-\beta).\tau_B(x)+\beta.(1-\min(\mu_B(x))))$

≥ 0,

because for every four real numbers a, b, c and d:

$\max(a, b) + \max(c, d) - \max(a + c, b + d) \geq 0$.

(k) $\bar{H}_{\alpha,\beta}(A + B)$

$= \bar{H}_{\alpha,\beta}(\{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x), \tau_A(x).\tau_B(x)\rangle / x \in E\})$

$= \{\langle x, \alpha.(\mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x)), \tau_A(x).\tau_B(x)+\beta.(1-\alpha.(\mu_A(x)+$

$\mu_B(x)-\mu_A(x).\mu_B(x))-\tau_A(x).\tau_B(x))\rangle / x \in E\}$

Page 4

- 6 -

(h) $\overline{J}_{\alpha,\beta}(\langle\rangle A) \subset \langle\rangle \overline{J}_{\alpha,\beta}(A)$,

Proof: (h) $\langle\rangle \overline{J}_{\alpha,\beta}(A) = \{\langle n, 1-\beta.\tau_A(x), \beta.\tau_A(x)\rangle / x \in E\}$

and

$\overline{J}_{\alpha,\beta}(\langle\rangle A) = \{\langle x, 1-\tau_A(x)+\alpha.(\tau_A(x)-\beta.\tau_A(x)), \beta.\tau_A(x)\rangle / x \in E\}$

From the equality of the third components of the two IFSs and from:

$1-\beta.\tau_A(x)-(1-\tau_A(x)+\alpha.(\tau_A(x)-\beta.\tau_A(x))$

$= \tau_A(x).(1-\alpha).(1-\beta) \geq 0$

follows the validity of (h).

The other assertions are proved analogically.

THEOREM 5: For every IFS A, for every $\alpha, \beta \in [0, 1]$ and for every $\gamma, \delta \in [0, 1]$ for which $0 \leq \gamma + \delta \leq 1$:

(a) $F_{\gamma,\delta}(H_{\alpha,\beta}(A)) \subset H_{\langle\rangle \alpha,\beta}(F_{\gamma,\delta}(A))$,

(b) $J_{\alpha,\beta}(F_{\gamma,\delta}(A)) \subset F_{[]\gamma,\delta}(J_{\alpha,\beta}(A))$.

Proof: (a) $F_{\gamma,\delta}(H_{\alpha,\beta}(A))$

$= \{\langle x, \alpha.\mu_A(x)+\gamma.(1-\alpha.\mu_A(x)-\tau_A(x)-\beta.\pi_A(x)), \tau_A(x)+\beta.\pi_A(x)+\delta.(1-\alpha.\mu_A(x)-\tau_A(x)-\beta.\pi_A(x))\rangle / x \in E\}$,

$H_{\alpha,\beta}(F_{\gamma,\delta}(A))$

$= \{\langle x, \alpha.(\mu_A(x)+\gamma.\pi_A(x)), \tau_A(x)+\delta.\pi_A(x)+\beta.(1-\mu_A(x)-\gamma.\pi_A(x)-\tau_A(x)-\delta.\pi_A(x))\rangle / x \in E\}$

From:

Page 5

- 6 -

(h) $\overline{J}_{\alpha,\beta}(\langle\rangle A) \subset \langle\rangle \overline{J}_{\alpha,\beta}(A)$,

Proof: (h) $\langle\rangle \overline{J}_{\alpha,\beta}(A) = \{\langle n, 1-\beta.\tau_A(x), \beta.\tau_A(x)\rangle / x \in E\}$

and

$\overline{J}_{\alpha,\beta}(\langle\rangle A) = \{\langle x, 1-\tau_A(x)+\alpha.(\tau_A(x)-\beta.\tau_A(x)), \beta.\tau_A(x)\rangle / x \in E\}$

From the equality of the third components of the two IFSs and from:

$1-\beta.\tau_A(x)-(1-\tau_A(x)+\alpha.(\tau_A(x)-\beta.\tau_A(x))$

$= \tau_A(x).(1-\alpha).(1-\beta) \geq 0$

follows the validity of (h).

The other assertions are proved analogically.

THEOREM 5: For every IFS A, for every $\alpha, \beta \in [0, 1]$ and for every $\gamma, \delta \in [0, 1]$ for which $0 \leq \gamma + \delta \leq 1$:

(a) $F_{\gamma,\delta}(H_{\alpha,\beta}(A)) \subset H_{\langle\rangle \alpha,\beta}(F_{\gamma,\delta}(A))$,

(b) $J_{\alpha,\beta}(F_{\gamma,\delta}(A)) \subset F_{[]\gamma,\delta}(J_{\alpha,\beta}(A))$.

Proof: (a) $F_{\gamma,\delta}(H_{\alpha,\beta}(A))$

$= \{\langle x, \alpha.\mu_A(x)+\gamma.(1-\alpha.\mu_A(x)-\tau_A(x)-\beta.\pi_A(x)), \tau_A(x)+\beta.\pi_A(x)+\delta.(1-\alpha.\mu_A(x)-\tau_A(x)-\beta.\pi_A(x))\rangle / x \in E\}$,

$H_{\alpha,\beta}(F_{\gamma,\delta}(A))$

$= \{\langle x, \alpha.(\mu_A(x)+\gamma.\pi_A(x)), \tau_A(x)+\delta.\pi_A(x)+\beta.(1-\mu_A(x)-\gamma.\pi_A(x)-\tau_A(x)-\delta.\pi_A(x))\rangle / x \in E\}$

From:

Page 6

- 7 -

$\beta.\pi_A(x)+\delta-\alpha.\delta.\mu_A(x)-\delta.\tau_A(x)-\beta-\delta.\pi_A(x)+\beta.\mu_A(x)+\beta.\gamma.\pi_A(x)+\beta.\tau_A(x)$

$= \delta.(1-\alpha.\mu_A(x)-\tau_A(x)-\pi_A(x))+\beta.\gamma.\pi_A(x)$

$\geq \delta.(1-\mu_A(x)-\tau_A(x)-\pi_A(x))+\beta.\gamma.\pi_A(x)$

$\geq \beta.\gamma.\pi_A(x) \geq 0$

follows the validity of (a).

(b) is proved analogically.

Other relations between the operators $F_{\gamma,\delta}$ (resp. D_α) and $H_{\alpha,\beta}$ and $J_{\alpha.\beta}$ are not valid.

THEOREM 6: For every IFS A, for every $\alpha, \beta, \gamma, \delta \in [0, 1]$:

(a) $H_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(H_{\alpha,\beta}(A))$

(b) $J_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(J_{\alpha,\beta}(A))$

(c) $\overline{H}_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(\overline{H}_{\alpha,\beta}(A))$

(d) $\overline{J}_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(\overline{J}_{\alpha,\beta}(A))$

Proof: (a) $H_{\alpha,\beta}(G_{\gamma,\delta}(A))$

$= \{\langle x, \alpha.\gamma.\mu_A(x), \delta.\tau_A(x)+\beta.(1-\gamma.\mu_A(x)-\delta.\tau_A(x))\rangle / x \in E\}$,

$G_{\gamma,\delta}(H_{\alpha,\beta}(A))$

$= \{\langle x, \alpha.\gamma.\mu_A(x), \delta.\tau_A(x)+\beta.\delta.(1-\mu_A(x)-\tau_A(x))\rangle / x \in E\}$.

The second terms in both sets are equal and for the third terms is valid the following inequality:

$\delta.\tau_A(x)+\beta.(1-\gamma.\mu_A(x)-\delta.\tau_A(x))-(\delta.\tau_A(x)+\beta.\delta.(1-\mu_A(x)-\tau_A(x)))$

$= \beta.(1-\gamma.\mu_A(x)-\delta.\tau_A(x)-\delta.(1-\mu_A(x)-\tau_A(x)))$

Page 7

- 8 -

$= \beta.(1-\gamma.\mu_A(x)-\delta+\delta.\mu_A(x))$

if $\gamma \geq \delta$:

$\geq \beta.(\gamma-\delta).(1-\mu_A(x)) \geq 0$;

if $\gamma < \delta$:

$\geq \beta.(1-\delta+\mu_A(x).(\delta-\gamma)) \geq 0$,

i.e. (a) is valid.

The other assertions are proved analogically.

REFERENCES:

[1] Atanassov K. Intuitionistic fuzzy sets, Fuzzy sets and Systems Vol. 20 (1986), No. 1, 87-96.

[2] Atanassov K. Review and new results on intuitionistic fuzzy sets. Preprint IM-MFAIS-1-88, Sofia, 1988.

[3] Atanassov K. More on intuitionistic fuzzy sets, Fuzzy sets and systems (in press).

[4] Atanassov K. Two operators on intuitionistic fuzzy sets, Comptes Rendus de l'Academie bulgare des Sciences, Tome 41, No. 5, 35-38.

Page 8