



## Four New Operators over Intuitionistic Fuzzy Sets

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FOUR NEW OPERATORS OVER INTUITIONISTIC FUZZY SETS  
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Over the object intuitionistic fuzzy set (IFS)  $A$  has been defined different operators  $D_\alpha$ ,  $C_\alpha$ ,  $R_\alpha$ ,  $I_\alpha$  ( $\alpha \in [0, 1]$ ) and others (see [1-4]). Here, by analogy with the operators  $D_\alpha$ ,  $F_{\alpha,\beta}$  and  $G_{\alpha,\beta}$ , we shall define four other operators for a given IFS  $A$  and for given numbers  $\alpha, \beta \in [0, 1]$ :

$H_{\alpha,\beta}(A) = (\{x, \alpha.\mu_A(x), v_A(x) + \beta.\pi_A(x)\} \mid x \in E)$

$\bar{H}_{\alpha,\beta}(A) = (\{x, \alpha.\mu_A(x), v_A(x) + \beta.(1 - \alpha.\mu_A(x) - v_A(x))\} \mid x \in E)$

$J_{\alpha,\beta}(A) = (\{x, \mu_A(x) + \alpha.\pi_A(x), \beta.v_A(x)\} \mid x \in E)$

$\bar{J}_{\alpha,\beta}(A) = (\{x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.v_A(x)), \beta.v_A(x)\} \mid x \in E)$

Below we shall introduce the basic assertions from these operators.

**THEOREM 1:** For every IFS  $A$  and for every  $\alpha, \beta \in [0, 1]$ :

(a)  $\overline{H_{\alpha,\beta}(\bar{A})} = J_{\beta,\alpha}(A)$ ,

(b)  $\overline{J_{\alpha,\beta}(\bar{A})} = H_{\beta,\alpha}(A)$ ,

(c)  $\overline{\bar{H}_{\alpha,\beta}(\bar{A})} = \bar{J}_{\beta,\alpha}(A)$ ,

(d)  $\overline{\bar{J}_{\alpha,\beta}(\bar{A})} = \bar{H}_{\beta,\alpha}(A)$ .

Over the object intuitionistic fuzzy set (IFS) [1, 2] has been defined different operators:  $\square$ ,  $\diamond$ ,  $D_\alpha$  ( $\alpha \in [0, 1]$ ),  $F_{\alpha,\beta}$  ( $\alpha, \beta \in [0, 1]$ ),  $G_{\alpha,\beta}$  ( $\alpha, \beta \in [0, 1]$ ), and others (see [1-4]). Here, by analogy with the operators  $D_\alpha$ ,  $F_{\alpha,\beta}$  and  $G_{\alpha,\beta}$ , we shall define four other operators for a given IFS  $A$  and for given numbers  $\alpha, \beta \in [0, 1]$ :

$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), v_A(x) + \beta.\pi_A(x) \rangle \mid x \in E\},$$

$$\bar{H}_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), v_A(x) + \beta.(1 - \alpha.\mu_A(x) - v_A(x)) \rangle \mid x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.v_A(x) \rangle \mid x \in E\},$$

$$\bar{J}_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.v_A(x)), \beta.v_A(x) \rangle \mid x \in E\}.$$

Below we shall introduce the basic assertions from these operators.

**Theorem 1:** For every IFS  $A$  and for every  $\alpha, \beta \in [0, 1]$ :

$$(a) \quad \overline{H_{\alpha,\beta}(\bar{A})} = J_{\beta,\alpha}(A),$$

$$(b) \quad \overline{J_{\alpha,\beta}(\bar{A})} = H_{\beta,\alpha}(A),$$

$$(c) \quad \overline{\bar{H}_{\alpha,\beta}(\bar{A})} = \bar{J}_{\beta,\alpha}(A),$$

$$(d) \quad \overline{\bar{J}_{\alpha,\beta}(\bar{A})} = \bar{H}_{\beta,\alpha}(A).$$

$$\begin{aligned} \text{Proof: (a)} \quad \overline{H_{\alpha,\beta}(\bar{A})} &= \{\langle x, \alpha.\mu_A(x), v_A(x) + \beta.\pi_A(x) \rangle \mid x \in E\} \\ &= \{\langle x, \mu_A(x) + \beta.\pi_A(x), \alpha.v_A(x) \rangle \mid x \in E\} \\ &= J_{\beta,\alpha}(A). \end{aligned}$$

(b)-(d) are proved analogically.  $\square$

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**Theorem 2:** For every IFS A and for every  $\alpha, \beta \in [0, 1]$ :

- (a)  $H_{\alpha,\beta}(A) = F_{0,\beta}(A) \cap G_{\alpha,1}(A)$ ,
- (b)  $J_{\alpha,\beta}(A) = F_{\beta,0}(A) \cup G_{1,\alpha}(A)$ ,
- (c)  $\bar{H}_{\alpha,\beta}(A) = F_{0,\beta}(G_{\alpha,1}(A))$ ,
- (d)  $\bar{H}_{\alpha,\beta}(A) = F_{0,\beta}(G_{\alpha,1}(A))$ .

*Proof:*

- (a)  $H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), v_A(x) + \beta.\pi_A(x) \rangle | x \in E\}$   
 $= \{\langle x, \min(\mu_A(x), \alpha.\mu_A(x)), \max(v_A(x) + \beta.\pi_A(x), v_A(x)) \rangle | x \in E\} = F_{0,\beta}(A) \cap G_{\alpha,1}(A)$ .
- (b) is proved analogically.
- (c)  $\bar{H}_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), v_A(x) + \beta.(1 - \alpha.\mu_A(x) - v_A(x)) \rangle | x \in E\}$   
 $= F_{0,\beta}(\{\langle x, \alpha.\mu_A(x), v_A(x) \rangle | x \in E\}) = F_{0,\beta}(G_{\alpha,1})$ .
- (d) is proved analogically. □

**Theorem 3:** For every IFSs A and B and for every  $\alpha, \beta \in [0, 1]$ :

- (a)  $H_{\alpha,\beta}(A \cap B) \subset H_{\alpha,\beta}(A) \cap H_{\alpha,\beta}(B)$ ,
- (b)  $H_{\alpha,\beta}(A \cup B) \supset H_{\alpha,\beta}(A) \cup H_{\alpha,\beta}(B)$ ,
- (c)  $H_{\alpha,\beta}(A + B) \subset H_{\alpha,\beta}(A) + H_{\alpha,\beta}(B)$ ,
- (d)  $H_{\alpha,\beta}(A.B) \supset H_{\alpha,\beta}(A).H_{\alpha,\beta}(B)$ ,
- (e)  $J_{\alpha,\beta}(A \cap B) \supset J_{\alpha,\beta}(A) \cap J_{\alpha,\beta}(B)$ ,
- (f)  $J_{\alpha,\beta}(A \cup B) \subset J_{\alpha,\beta}(A) \cup J_{\alpha,\beta}(B)$ ,
- (g)  $J_{\alpha,\beta}(A + B) \supset J_{\alpha,\beta}(A) + J_{\alpha,\beta}(B)$ ,
- (h)  $J_{\alpha,\beta}(A.B) \subset J_{\alpha,\beta}(A).J_{\alpha,\beta}(B)$ ,
- (i)  $\bar{H}_{\alpha,\beta}(A \cap B) \subset \bar{H}_{\alpha,\beta}(A) \cap \bar{H}_{\alpha,\beta}(B)$ ,
- (j)  $\bar{H}_{\alpha,\beta}(A \cup B) \supset \bar{H}_{\alpha,\beta}(A) \cup \bar{H}_{\alpha,\beta}(B)$ ,
- (k)  $\bar{H}_{\alpha,\beta}(A + B) \subset \bar{H}_{\alpha,\beta}(A) + \bar{H}_{\alpha,\beta}(B)$ ,
- (l)  $\bar{H}_{\alpha,\beta}(A.B) \supset \bar{H}_{\alpha,\beta}(A).\bar{H}_{\alpha,\beta}(B)$ ,
- (m)  $\bar{J}_{\alpha,\beta}(A \cap B) \supset \bar{J}_{\alpha,\beta}(A) \cap \bar{J}_{\alpha,\beta}(B)$ ,
- (n)  $\bar{J}_{\alpha,\beta}(A \cup B) \subset \bar{J}_{\alpha,\beta}(A) \cup \bar{J}_{\alpha,\beta}(B)$ ,

- (o)  $\bar{J}_{\alpha,\beta}(A+B) \supseteq \bar{J}_{\alpha,\beta}(A) + \bar{J}_{\alpha,\beta}(B)$ ,
- (p)  $\bar{J}_{\alpha,\beta}(A \cdot B) \subset \bar{J}_{\alpha,\beta}(A) \cdot \bar{J}_{\alpha,\beta}(B)$ .

*Proof:*

$$\begin{aligned}
 (a) \quad H_{\alpha,\beta}(A \cap B) &= H_{\alpha,\beta}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(v_A(x), v_B(x)) \rangle | x \in E\}) \\
 &= \{\langle x, \alpha \cdot \min(\mu_A(x), \mu_B(x)), \max(v_A(x), v_B(x)) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x)) - \max(v_A(x), v_B(x))) \rangle | x \in E\} \\
 H_{\alpha,\beta}(A) \cap H_{\alpha,\beta}(B) &= \{\langle x, \alpha \cdot \mu_A(x), v_A(x) + \beta \cdot \pi_A(x) \rangle | x \in E\} \cap \{\langle x, \alpha \cdot \mu_B(x), v_B(x) + \beta \cdot \pi_B(x) \rangle | x \in E\} \\
 &= \{\langle x, \min(\alpha \cdot \mu_A(x), \alpha \cdot \mu_B(x)), \max(v_A(x) + \beta \cdot (1 - \min(\mu_A(x) - v_A(x), v_B(x) + \beta \cdot (1 - \min(\mu_B(x) - v_B(x)))) | x \in E\}
 \end{aligned}$$

The second terms (the degrees of membership) in both sets are equal and for the third terms is valid the following inequality:

$$\begin{aligned}
 &\max(v_A(x), v_B(x)) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x)) - \max(v_A(x), v_B(x))) \\
 &- \max(v_A(x) + \beta \cdot (1 - \min(\mu_A(x) - v_A(x), v_B(x) + \beta \cdot (1 - \min(\mu_B(x) - v_B(x)))) \\
 &= (1 - \beta) \cdot \max(v_A(x), v_B(x)) + \beta \cdot \max(1 - (\mu_A(x), 1 - \mu_B(x)) - \max((1 - \beta) \cdot v_A(x) + \beta \cdot (1 - \min(\mu_A(x))), (1 - \beta) \cdot v_B(x) + \beta \cdot (1 - \min(\mu_B(x)))) \geq 0,
 \end{aligned}$$

because for every real numbers  $a, b, c$  and  $d$ :

$$\max(a, b) + \max(c, d) - \max(a + c, b + d) \geq 0.$$

$$\begin{aligned}
 (k) \quad \bar{H}_{\alpha,\beta}(A+B) &= \bar{H}_{\alpha,\beta}(\{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), v_A(x) \cdot v_B(x) \rangle | x \in E\}) \\
 &= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), v_A(x) \cdot v_B(x) + \beta \cdot (1 - \alpha \cdot (\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)) - v_A(x) \cdot v_B(x)) \rangle | x \in E\}.
 \end{aligned}$$

$$\begin{aligned}
 H_{\alpha,\beta}(A) + H_{\alpha,\beta}(B) &= \{\langle x, \mu_A(x), v_A(x) + \beta \cdot \pi_A(x) \rangle | x \in E\} + \{\langle x, \mu_B(x), v_B(x) + \beta \cdot \pi_B(x) \rangle | x \in E\} \\
 &= \{\langle x, \alpha \cdot \mu_A(x) + \alpha \cdot \mu_B(x) - \alpha^2 \cdot \mu_A(x) \cdot \mu_B(x), (v_A(x) + \beta \cdot \pi_A(x)) \cdot (v_B(x) + \beta \cdot \pi_B(x)) \rangle | x \in E\}.
 \end{aligned}$$

Obviously the second term of the first set is greater than the second term of the second set. On the other hand:

$$\begin{aligned}
 &(v_A(x) + \beta \cdot \pi_A(x)) \cdot (v_B(x) + \beta \cdot \pi_B(x)) - v_A(x) \cdot v_B(x) + \beta \cdot (1 - \alpha \cdot (\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)) - v_A(x) \cdot v_B(x)) \\
 &\geq \beta \cdot (1 - \alpha \cdot (\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) - v_A(x) \cdot v_B(x))) \\
 &\geq \beta \cdot (1 - \mu_A(x) - \mu_B(x) + \mu_A(x) \cdot \mu_B(x) - v_A(x) \cdot v_B(x))) \\
 &\geq 0.
 \end{aligned}$$

The other assertions are proved analogically.  $\square$

**Theorem 4:** For every IFS  $A$  and for every  $\alpha, \beta \in [0, 1]$ :

- (a)  $\square H_{\alpha,\beta}(A) \subset H_{\alpha,\beta}(\square A)$ ,
- (b)  $\diamond H_{\alpha,\beta}(A) \subset_{\diamond} H_{\alpha,\beta}(\diamond A)$ ,
- (c)  $J_{\alpha,\beta}(\square A) \subset_{\square} \square J_{\alpha,\beta}(A)$ ,
- (d)  $J_{\alpha,\beta}(\diamond A) \subset \diamond J_{\alpha,\beta}(A)$ ,
- (e)  $\square \bar{H}_{\alpha,\beta}(A) \subset \bar{H}_{\alpha,\beta}(\square A)$ ,
- (f)  $\diamond \bar{H}_{\alpha,\beta}(A) \subset_{\diamond} \bar{H}_{\alpha,\beta}(\diamond A)$ ,
- (g)  $\bar{J}_{\alpha,\beta}(\square A) \subset_{\square} \square \bar{J}_{\alpha,\beta}(A)$ ,
- (h)  $\bar{J}_{\alpha,\beta}(\diamond A) \subset \diamond \bar{J}_{\alpha,\beta}(A)$ .

*Proof:*

$$(h) \quad \begin{aligned} \bar{J}_{\alpha,\beta}(\diamond A) &= \{\langle x, 1 - \beta \cdot v_A(x), \beta \cdot v_A(x) \rangle \mid x \in E\} \text{ and} \\ \bar{J}_{\alpha,\beta}(A) &= \{\langle x, 1 - v_A(x) + \alpha \cdot (v_A(x) - \beta \cdot v_A(x)), \beta \cdot v_A(x) \rangle \mid x \in E\}. \end{aligned}$$

From the equality of the third components of the two IFSs and from

$$1 - \beta \cdot v_A(x) - (1 - v_A(x) + \alpha \cdot (v_A(x) - \beta \cdot v_A(x))) = v_A(x) \cdot (1 - \alpha) \cdot (1 - \beta) \geq 0$$

follows the validity of (h).

The other assertions are proved analogically.  $\square$

**Theorem 5:** For every IFS  $A$  and for every  $\alpha, \beta \in [0, 1]$  and for every  $\gamma, \delta \in [0, 1]$  for which  $0 \leq \gamma + \delta \leq 1$ :

- (a)  $F_{\gamma,\delta}(H_{\alpha,\beta}(A)) \subset_{\diamond} H_{\alpha,\beta}(F_{\gamma,\delta}(A))$ ,
- (b)  $J_{\alpha,\beta}(F_{\gamma,\delta}(A)) \subset_{[]} F_{\gamma,\delta}(J_{\alpha,\beta}(A))$ .

*Proof:*

$$\begin{aligned} (a) \quad &F_{\gamma,\delta}(H_{\alpha,\beta}(A)) \\ &= \{\langle x, \alpha \cdot \mu_A(x) + \gamma \cdot (1 - \alpha \cdot \mu_A(x) - v_A(x) - \beta \cdot \pi_A(x)), v_A(x) + \beta \cdot \pi_A(x) + \delta \cdot (1 - \alpha \cdot \mu_A(x) \\ &\quad - v_A(x) - \beta \cdot \pi_A(x)) \rangle \mid x \in E\}, \\ &H_{\alpha,\beta}(F_{\gamma,\delta}(A)) \\ &= \{\langle x, \alpha \cdot (\mu_A(x) + \gamma \cdot \pi_A(x)), v_A(x) + \delta \cdot \pi_A(x) + \beta \cdot (1 - \mu_A(x) - \gamma \cdot \pi_A(x) - v_A(x) \\ &\quad - \delta \cdot \pi_A(x)) \rangle \mid x \in E\}. \end{aligned}$$

From:

$$\begin{aligned}
& \beta.\pi_A(x) + \delta - \alpha.\delta.\mu_A(x) - \delta.v_A(x) - \beta - \delta.\pi_A(x) + \beta.\mu_A(x) + \beta.\gamma.\pi_A(x) + \beta.v_A(x) \\
&= \delta.(1 - \alpha.\mu_A(x) - v_A(x) - \pi_A(x)) + \beta.\gamma.\pi_A(x) \\
&\geq \delta.(1 - \mu_A(x) - v_A(x) - \pi_A(x)) + \beta.\gamma.\pi_A(x) \\
&\geq \beta.\gamma.\pi_A(x) \geq 0
\end{aligned}$$

follows the validity of (a).

(b) is proved analogically.  $\square$

Other relations between the operators  $F_{\gamma,\delta}$  (resp.  $D_\alpha$ ) and  $H_{\alpha,\beta}$  and  $J_{\alpha,\beta}$  are not valid.

**Theorem 6:** For every IFS  $A$  and for every  $\alpha, \beta, \gamma, \delta \in [0, 1]$ :

- (a)  $H_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(H_{\alpha,\beta}(A)),$
- (b)  $J_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(J_{\alpha,\beta}(A)),$
- (c)  $\bar{H}_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(\bar{H}_{\alpha,\beta}(A)),$
- (d)  $\bar{J}_{\alpha,\beta}(G_{\gamma,\delta}(A)) \subset G_{\gamma,\delta}(\bar{J}_{\alpha,\beta}(A)).$

*Proof:*

$$\begin{aligned}
(a) \quad & H_{\alpha,\beta}(G_{\gamma,\delta}(A)) \\
&= \{\langle x, \alpha.\gamma.\mu_A(x), \delta.v_A(x) + \beta.(1 - \gamma.\mu_A(x) - \delta.v_A(x)) \rangle \mid x \in E\}, \\
(G_{\gamma,\delta}(H_{\alpha,\beta}(A))) &= \{\langle x, \alpha.\gamma.\mu_A(x), \delta.v_A(x) + \beta.\delta.(1 - \mu_A(x) - v_A(x)) \rangle \mid x \in E\}.
\end{aligned}$$

The second terms in both sets are equal and for the third terms is valid the following inequality:  
 $\delta.v_A(x) + \beta.(1 - \gamma.\mu_A(x) - \delta.v_A(x)) - (\delta.v_A(x) + \beta.\delta.(1 - \mu_A(x) - v_A(x)))$   
 $= \beta.(1 - \gamma.\mu_A(x) - \delta.v_A(x) - \delta.(1 - \mu_A(x) - v_A(x)))$   
 $= \beta.(1 - \gamma.\mu_A(x) - \delta + \delta.\mu_A(x))$

$$\begin{aligned}
& \text{if } \gamma \geq \delta : \\
& \geq \beta.(\gamma - \delta).(1 - \mu_A(x)) \geq 0;
\end{aligned}$$

$$\begin{aligned}
& \text{if } \gamma < \delta : \\
& \geq \beta.(1 - \delta + \mu_A(x).(\delta - \gamma)) \geq 0;
\end{aligned}$$

i.e., (a) is valid.

The other assertions are proved analogically.  $\square$

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Over the object intuitionistic fuzzy set (IFS) [1], [2] has been defined different operators:  $\sqcap$ ,  $\sqcup$ ,  $D_{\alpha, \beta}$  ( $\alpha \in [0, 1]$ ),  $F_{\alpha, \beta}$  ( $\alpha, \beta \in [0, 1]$ ),  $G_{\alpha, \beta}$  ( $\alpha, \beta \in [0, 1]$ ) and others (see [1]-[4]). Here, by analogy with the operators  $D_{\alpha, \beta}$ ,  $F_{\alpha, \beta}$  and  $G_{\alpha, \beta}$ , we shall define four other operators for a given IFS  $A$  and for given numbers  $\alpha, \beta \in [0, 1]$ :

$H_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x), \tau_A(x) + \beta \cdot \pi_A(x) \rangle / x \in E \rangle$ ,

$J_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x), \tau_A(x) + \beta \cdot (1 - \mu_A(x) - \pi_A(x)) \rangle / x \in E \rangle$ ,

$J_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x) + \alpha \cdot \pi_A(x), \beta \cdot \tau_A(x) \rangle / x \in E \rangle$ ,

$J_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x) + \alpha \cdot (1 - \mu_A(x) - \pi_A(x)), \beta \cdot \tau_A(x) \rangle / x \in E \rangle$ .

Below we shall introduce the basic assertions from these operators.

**THEOREM 1:** For every IFS  $A$  and for every  $\alpha, \beta \in [0, 1]$ :

(a)  $H_{\alpha, \beta}(A) = J_{\beta, \alpha}(A)$ ,

(b)  $J_{\alpha, \beta}(A) = H_{\beta, \alpha}(A)$ ,

(c)  $H_{\alpha, \beta}(A) = J_{\alpha, \beta}(A)$ ,

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$$(d) J_{\alpha, \beta}(A) = H_{\beta, \alpha}(A).$$

$$\text{Proof: (a)} H_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x), \mu_A(x) + \beta \cdot \pi_A(x) \rangle / x \in E \rangle \\ = \langle \langle x, \mu_A(x) + \beta \cdot \pi_A(x), \alpha \cdot \pi_A(x) \rangle / x \in E \rangle = J_{\beta, \alpha}(A).$$

(b)-(d) are proved analogically.

**THEOREM 2:** For every IFS  $A$  and for every  $\alpha, \beta \in [0, 1]$ :

$$(a) H_{\alpha, \beta}(A) = F_{\alpha, \beta}(A) \sqcap G_{\alpha, \beta}(A),$$

$$(b) J_{\alpha, \beta}(A) = F_{\alpha, \beta}(A) \sqcup G_{\alpha, \beta}(A),$$

$$(c) H_{\alpha, \beta}(A) = F_{\alpha, \beta}(G_{\alpha, \beta}(A)),$$

$$(d) J_{\alpha, \beta}(A) = F_{\alpha, \beta}(G_{\alpha, \beta}(A)).$$

$$\text{Proof: (a)} H_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x), \tau_A(x) + \beta \cdot \pi_A(x) \rangle / x \in E \rangle \\ = \langle \langle x, \min(\mu_A(x), \mu_A(x)), \max(\tau_A(x) + \beta \cdot \pi_A(x), \tau_A(x)) \rangle / x \in E \rangle$$

$$= F_{\alpha, \beta}(G_{\alpha, \beta}(A)).$$

(b) is proved analogically.

$$(c) H_{\alpha, \beta}(A) = \langle \langle x, \mu_A(x), \tau_A(x) + \beta \cdot (1 - \mu_A(x) - \pi_A(x)) \rangle / x \in E \rangle \\ = F_{\alpha, \beta}(\langle \langle x, \mu_A(x), \tau_A(x) \rangle / x \in E \rangle) \\ = F_{\alpha, \beta}(G_{\alpha, \beta}(A)).$$

(d) is proved analogically.

**THEOREM 3:** For every two IFSs  $A$  and  $B$  and for every  $\alpha, \beta \in [0, 1]$ :

$$(a) H_{\alpha, \beta}(A \sqcap B) \subseteq H_{\alpha, \beta}(A) \sqcap H_{\alpha, \beta}(B),$$

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- (b)  $H_{\alpha, \beta}(A \sqcup B) \supseteq H_{\alpha, \beta}(A) \sqcup H_{\alpha, \beta}(B)$ ,
  - (c)  $H_{\alpha, \beta}(A + B) \subset H_{\alpha, \beta}(A) + H_{\alpha, \beta}(B)$ ,
  - (d)  $H_{\alpha, \beta}(A \cdot B) \supseteq H_{\alpha, \beta}(A) \cdot H_{\alpha, \beta}(B)$ ,
  - (e)  $J_{\alpha, \beta}(A \sqcap B) \supseteq J_{\alpha, \beta}(A) \sqcap J_{\alpha, \beta}(B)$ ,
  - (f)  $J_{\alpha, \beta}(A \sqcup B) \subset J_{\alpha, \beta}(A) \sqcup J_{\alpha, \beta}(B)$ ,
  - (g)  $J_{\alpha, \beta}(A + B) \supseteq J_{\alpha, \beta}(A) + J_{\alpha, \beta}(B)$ ,
  - (h)  $J_{\alpha, \beta}(A \cdot B) \subset J_{\alpha, \beta}(A) \cdot J_{\alpha, \beta}(B)$ ,
  - (i)  $H_{\alpha, \beta}(A \sqcap B) \subset H_{\alpha, \beta}(A) \sqcap H_{\alpha, \beta}(B)$ ,
  - (j)  $H_{\alpha, \beta}(A \sqcup B) \supseteq H_{\alpha, \beta}(A) \sqcup H_{\alpha, \beta}(B)$ ,
  - (k)  $H_{\alpha, \beta}(A + B) \subset H_{\alpha, \beta}(A) + H_{\alpha, \beta}(B)$ ,
  - (l)  $H_{\alpha, \beta}(A \cdot B) \supseteq H_{\alpha, \beta}(A) \cdot H_{\alpha, \beta}(B)$ ,
  - (m)  $J_{\alpha, \beta}(A \sqcap B) \supseteq J_{\alpha, \beta}(A) \sqcap J_{\alpha, \beta}(B)$ ,
  - (n)  $J_{\alpha, \beta}(A \sqcup B) \subset J_{\alpha, \beta}(A) \sqcup J_{\alpha, \beta}(B)$ ,
  - (o)  $J_{\alpha, \beta}(A + B) \supseteq J_{\alpha, \beta}(A) + J_{\alpha, \beta}(B)$ ,
  - (p)  $J_{\alpha, \beta}(A \cdot B) \subset J_{\alpha, \beta}(A) \cdot J_{\alpha, \beta}(B)$ .
- Proof: (a)  $H_{\alpha, \beta}(A \sqcap B)$
- $$= H_{\alpha, \beta}(\langle \langle x, \min(\mu_A(x), \mu_B(x)), \max(\tau_A(x), \tau_B(x)) \rangle / x \in E \rangle)$$

$$= \langle \langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \tau_A(x) \cdot \tau_B(x) \rangle / x \in E \rangle$$

$$= H_{\alpha, \beta}(\langle \langle x, \min(\mu_A(x), \mu_B(x)), \max(\tau_A(x), \tau_B(x)) \rangle / x \in E \rangle)$$

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$$= \langle \langle x, \alpha \cdot \min(\mu_A(x), \mu_B(x)), \max(\tau_A(x), \tau_B(x)) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot \max(\tau_A(x), \tau_B(x)) - \max(\tau_A(x), \tau_B(x)) \rangle / x \in E \rangle$$

$$H_{\alpha, \beta}(A) \sqcap H_{\alpha, \beta}(B)$$

$$= \langle \langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \tau_A(x) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot \max(\tau_A(x), \tau_B(x)) - \max(\tau_A(x), \tau_B(x)) \rangle / x \in E \rangle$$

$$= \langle \langle x, \min(\alpha \cdot \mu_A(x), \alpha \cdot \mu_B(x)), \max(\tau_A(x) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot \max(\tau_A(x), \tau_B(x)) - \max(\tau_A(x), \tau_B(x)) \rangle / x \in E \rangle$$

The second terms (the degrees of membership) in both sets are equal and for the third terms is valid the following inequality:

$$\max(\tau_A(x), \tau_B(x)) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot \max(\tau_A(x), \tau_B(x)) - \max(\tau_A(x), \tau_B(x))$$

$$= \max(\tau_A(x) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot \max(\tau_A(x), \tau_B(x)) - \max(\tau_A(x), \tau_B(x)))$$

$$= (1 - \beta) \cdot \max(\tau_A(x), \tau_B(x)) + \beta \cdot \max(1 - \min(\mu_A(x), \mu_B(x)), 1 - \max(\tau_A(x), \tau_B(x)))$$

$$+ \tau_A(x) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot (1 - \max(\tau_A(x), \tau_B(x)))$$

$$\geq 0,$$

because for every four real numbers  $a, b, c$  and  $d$ :

$$\max(a, b) + \max(c, d) - \max(a + c, b + d) \geq 0.$$

$$(k) H_{\alpha, \beta}(A + B)$$

$$= H_{\alpha, \beta}(\langle \langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \tau_A(x) \cdot \tau_B(x) \rangle / x \in E \rangle)$$

$$= \langle \langle x, \alpha \cdot (\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)), \max(\tau_A(x) + \beta \cdot (1 - \min(\mu_A(x), \mu_B(x))) \cdot \max(\tau_A(x), \tau_B(x)) - \max(\tau_A(x), \tau_B(x)) \rangle / x \in E \rangle$$

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$$(h) \bar{J}_{\alpha, \beta} ((\wedge A)) \subset \bar{J}_{\alpha, \beta} (A)$$

Proof: (h)  $\bar{J}_{\alpha, \beta} (A) = \langle \langle n, 1-\beta \cdot \tau(x), \beta \cdot \tau(x) \rangle / x \in E \rangle$

and

$$\bar{J}_{\alpha, \beta} ((\wedge A)) = \langle \langle x, 1-\tau(x)+\alpha \cdot (\tau(x)-\beta \cdot \tau(x)), \beta \cdot \tau(x) \rangle / x \in E \rangle$$

From the equality of the third components of the two IFSs and

from:

$$1-\beta \cdot \tau(x)-(1-\tau(x))+\alpha \cdot (\tau(x)-\beta \cdot \tau(x))$$

$$= \tau(x) \cdot (1-\alpha) \cdot (1-\beta) \geq 0$$

follows the validity of (h).

The other assertions are proved analogically.

THEOREM 5: For every IFS A, for every  $\alpha, \beta \in [0, 1]$  and for every

$$\Gamma, \delta \in [0, 1]$$
 for which  $0 \leq \Gamma + \delta \leq 1$ :

$$(a) F_{\Gamma, \delta} (H_{\alpha, \beta} (A)) \subset H_{\Gamma, \delta} (F_{\alpha, \beta} (A))$$

$$(b) J_{\alpha, \beta} (F_{\Gamma, \delta} (A)) \subset F_{\Gamma, \delta} (J_{\alpha, \beta} (A))$$

Proof: (a)  $F_{\Gamma, \delta} (H_{\alpha, \beta} (A))$ 

$$= \langle \langle x, \alpha \cdot \mu(x) + \Gamma \cdot (1-\alpha) \cdot \mu(x) - \tau(x) - \beta \cdot \pi(x), \tau(x) + \beta \cdot \pi(x) + \delta \cdot (1-\alpha) \cdot \mu(x) - \tau(x) - \beta \cdot \pi(x) \rangle / x \in E \rangle$$

$$= \langle \langle x, \alpha \cdot (\mu(x) + \Gamma \cdot \pi(x)), \tau(x) + \beta \cdot \pi(x) + \beta \cdot (1-\mu(x)) - \Gamma \cdot \pi(x) - \tau(x) - \beta \cdot \pi(x) \rangle / x \in E \rangle$$

From:

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$$(h) \bar{J}_{\alpha, \beta} ((\wedge A)) \subset \bar{J}_{\alpha, \beta} (A)$$

Proof: (h)  $\bar{J}_{\alpha, \beta} (A) = \langle \langle n, 1-\beta \cdot \tau(x), \beta \cdot \tau(x) \rangle / x \in E \rangle$

and

$$\bar{J}_{\alpha, \beta} ((\wedge A)) = \langle \langle x, 1-\tau(x)+\alpha \cdot (\tau(x)-\beta \cdot \tau(x)), \beta \cdot \tau(x) \rangle / x \in E \rangle$$

From the equality of the third components of the two IFSs and

from:

$$1-\beta \cdot \tau(x)-(1-\tau(x))+\alpha \cdot (\tau(x)-\beta \cdot \tau(x))$$

$$= \tau(x) \cdot (1-\alpha) \cdot (1-\beta) \geq 0$$

follows the validity of (h).

The other assertions are proved analogically.

THEOREM 5: For every IFS A, for every  $\alpha, \beta \in [0, 1]$  and for every

$$\Gamma, \delta \in [0, 1]$$
 for which  $0 \leq \Gamma + \delta \leq 1$ :

$$(a) F_{\Gamma, \delta} (H_{\alpha, \beta} (A)) \subset H_{\Gamma, \delta} (F_{\alpha, \beta} (A))$$

$$(b) J_{\alpha, \beta} (F_{\Gamma, \delta} (A)) \subset F_{\Gamma, \delta} (J_{\alpha, \beta} (A))$$

Proof: (a)  $F_{\Gamma, \delta} (H_{\alpha, \beta} (A))$ 

$$= \langle \langle x, \alpha \cdot \mu(x) + \Gamma \cdot (1-\alpha) \cdot \mu(x) - \tau(x) - \beta \cdot \pi(x), \tau(x) + \beta \cdot \pi(x) + \delta \cdot (1-\alpha) \cdot \mu(x) - \tau(x) - \beta \cdot \pi(x) \rangle / x \in E \rangle$$

$$= \langle \langle x, \alpha \cdot (\mu(x) + \Gamma \cdot \pi(x)), \tau(x) + \beta \cdot \pi(x) + \beta \cdot (1-\mu(x)) - \Gamma \cdot \pi(x) - \tau(x) - \beta \cdot \pi(x) \rangle / x \in E \rangle$$

From:

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$$\beta \cdot \pi(x) + \delta - \alpha \cdot \delta \cdot \mu(x) - \delta \cdot \tau(x) - \beta - \delta \cdot \pi(x) + \beta \cdot \mu(x) + \beta \cdot \Gamma \cdot \pi(x) + \beta \cdot \tau(x)$$

$$= \delta \cdot (1 - \alpha \cdot \mu(x) - \tau(x) - \pi(x)) + \beta \cdot \Gamma \cdot \pi(x)$$

$$\geq \delta \cdot (1 - \mu(x) - \tau(x) - \pi(x)) + \beta \cdot \Gamma \cdot \pi(x)$$

$$\geq \beta \cdot \Gamma \cdot \pi(x) \geq 0$$

follows the validity of (a).

(b) is proved analogically.

Other relations between the operators  $F_{\Gamma, \delta}$  (resp.  $D_{\alpha, \beta}$ ) and  $H_{\alpha, \beta}$  and  $J_{\alpha, \beta}$  are not valid.THEOREM 6: For every IFS A, for every  $\alpha, \beta, \Gamma, \delta \in [0, 1]$ :

$$(a) H_{\Gamma, \delta} (G_{\alpha, \beta} (A)) \subset G_{\Gamma, \delta} (H_{\alpha, \beta} (A))$$

$$(b) J_{\alpha, \beta} (G_{\Gamma, \delta} (A)) \subset G_{\Gamma, \delta} (J_{\alpha, \beta} (A))$$

$$(c) H_{\Gamma, \delta} (G_{\alpha, \beta} (A)) \subset G_{\Gamma, \delta} (H_{\alpha, \beta} (A))$$

$$(d) J_{\alpha, \beta} (G_{\Gamma, \delta} (A)) \subset G_{\Gamma, \delta} (J_{\alpha, \beta} (A))$$

Proof: (a)  $H_{\Gamma, \delta} (G_{\alpha, \beta} (A))$ 

$$= \langle \langle x, \alpha \cdot \Gamma \cdot \mu(x), \delta \cdot \tau(x) + \beta \cdot (1-\Gamma \cdot \mu(x)) - \delta \cdot \tau(x) \rangle / x \in E \rangle$$

$$= \langle \langle x, \alpha \cdot \Gamma \cdot \mu(x), \delta \cdot \tau(x) + \beta \cdot \delta \cdot (1-\mu(x)) - \delta \cdot \tau(x) \rangle / x \in E \rangle$$

$$= \langle \langle x, \alpha \cdot \Gamma \cdot \mu(x), \delta \cdot \tau(x) + \beta \cdot \delta \cdot (1-\mu(x)) - \delta \cdot \tau(x) \rangle / x \in E \rangle$$

The second terms in both sets are equal and for the third terms

is valid the following inequality:

$$\delta \cdot \tau(x) + \beta \cdot (1-\Gamma \cdot \mu(x)) - \delta \cdot \tau(x) - \beta \cdot \delta \cdot (1-\mu(x)) + \beta \cdot \delta \cdot (1-\mu(x)) - \delta \cdot \tau(x)$$

$$= \beta \cdot (1-\Gamma \cdot \mu(x)) - \delta \cdot \tau(x) - \beta \cdot (1-\mu(x)) + \beta \cdot \delta \cdot (1-\mu(x))$$

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$$= \beta \cdot (1-\Gamma \cdot \mu(x)) - \delta + \delta \cdot \mu(x)$$

if  $\Gamma \geq \delta$ :

$$\geq \beta \cdot (\Gamma - \delta) \cdot (1 - \mu(x)) \geq 0$$

if  $\Gamma < \delta$ :

$$\geq \beta \cdot (1 - \delta + \mu(x)) \cdot (\delta - \Gamma) \geq 0$$

i.e. (a) is valid.

The other assertions are proved analogically.

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