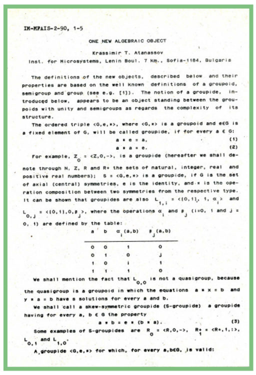
One New Algebraic Object

Krassimir T. Atanassov*



How to cite:

Atanassov K. T. One New Algebraic Object, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1990, Preprint IM-MFAIS-2-90. Reprinted: Int J Bioautomation, 2016, 20(S1), S75-S81.

The definitions of the new objects, described below and their properties are based on the well-known definitions of a groupoid, semigroup and group (see e.g. [1]). The notion of a ***groupide***, introduced below, appears to be an object standing between the groupoids with unity and semigroups as regards the complexity of its structure.

The ordered triple $\langle G, e, *\rangle$, where $\langle G, *\rangle$ is a groupoid and $e \in G$ is a fixed element of G, will be called ***groupide***, if for every $a \in G$:

$$a * e = a, \tag{1}$$

$$a * a = e, \tag{2}$$

For example, $Z_0 = \langle Z, 0, -\rangle$ is a groupide (hereafter we shall denote through N, Z, R and R^+ the sets of natural, integer, real and positive real numbers); $S = \langle G, e, *\rangle$ is a groupide, if G is the set of axial (central) symmetries, e is the identity, and $*$ is the operation composition between two symmetries from the respective type. It can be shown that groupides are also $L_{1,i} = \langle\{0, 1\}, 1, \alpha_i\rangle$ and $L_{0,j} = \langle\{0, 1\}, 0, \beta_j\rangle$ where the operations α_i and β_j ($i = 0, 1$ and $j = 0, 1$) are defined by the table:

a	b	$\alpha_i(a, b)$	$\beta_j(a, b)$
0	0	1	0
0	1	0	j
1	0	i	1
1	1	1	0

We shall mention the fact that $L_{0,0}$ is not a quasigroup; the quasigroup is a groupoid in which the equations $a * x = b$ and $y * a = b$ have solutions for every a and b.

We shall call a skew-symmetric groupide (***S*-groupide**) a groupide having for every $a, b \in G$ the property

$$a * b = e * (b * a). \tag{3}$$

Some examples of S-groupides are $R_0 = \langle R, 0, -\rangle$, $R_1^+ = \langle R^+, 1, :\rangle$, $L_{0,0}$ and $L_{1,0}$.

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A groupide $\langle G, e, *\rangle$ for which,for every $a, b \in G$ is valid:

$$(a*b)*c = a*(b*(e*c)) \tag{4}$$

will be called L-groupide. R_0, R_1^+, $L_{0,1}$ and $L_{1,0}$ are L-groupide.

Let $\langle G, e, *\rangle$ be L-groupide. We define for every $a, b \in G$ a new operation @ through:

$$a*b = c \text{ iff } a = c@b. \tag{5}$$

From (2) and (5) it follows directly that for every $a \in G$:

$$e@a = a. \tag{6}$$

Theorem 1: If the equivalence (5) is valid and $\langle G, e, *\rangle$ is a L-groupide , the $\langle G, e, @\rangle$ is a left group.

Proof. Let everywhere below a and b are arbitrary elements of G. Initially we shall mention that:

$$a*b = e \text{ iff } a = b. \tag{7}$$

Indeed, from (5) and (6) it follows directly:

$$a*b = e \text{ iff } a = e@b \text{ iff } a = b.$$

Also from (5) the equation follows directly:

$$(a*b)@b = a. \tag{8}$$

$$(a@b)*b = a. \tag{9}$$

We shall check the validity of the equation:

$$a@b = b*(e*a). \tag{10}$$

Sequentially we obtain from (4), (9) and (2):

$$(a@b)*(b*(e*a)) = ((a*b)*b*a = a*a = e$$

and the validity of (10) follows from (7).

Let $a, b, c \in G$ and let $(a@b)@c = x$. Then $a@b = x*c$. But from (9), (5), (4) and (10):

$$a = (a@b)*b = (x*c)*b = x*(c*(e*b)) = x*(b@c)$$

Then we derive from (8):

$$a@(b@c) = (x*(b@c))@(b@c) = x$$

i.e.,

$$(a@b)@c = a@(b@c).$$

Hence, the operation @ is associative over G. We shall show that for every $a \in G$ there exists unique $b \in G$ for which:

$b@a = e.$

Let $b = e * a$. Then from (8):

$b@a = (e * a)@a = e.$

Let for a fixed $a \in G : b@a = e$ and $c@a = e$. Then:

$b * c = (e * a) * (e * a) = e$

and from (7) it follows $b = c$, i.e. $\langle G, e, @\rangle$ is a left group. Because $a@e = a$ not always in the frames of the left group $\langle G, e, @\rangle$, the Eq. (1) is not always derivable. From this it follows that it is not always possible to prove that if $\langle G, e, @\rangle$ is a left group and (5) is valid, then $\langle G, e, *\rangle$ is a L-groupide. We shall prove also for every $a, b, c \in G$ the following equation:

$$a * (b * c) = (a@c) * b. \tag{11}$$

Let $f = a * (b * c)$. Hence $f * (b * c) = a$. Then, from the associativity in $\langle G, e, @\rangle$, and from (8) and (9) it follows that:

$(a@c) * b = ((f@(b * c)@c) * b + (f@((b * c)@c) * b = (f@b) * b = f.$

In the same way for every $a, b, c \in G$ the following equations can be proved:

$$a@(b * c) = (a@b) * c \tag{12}$$
$$a * (b@c) = (a * c) * b \tag{13}$$
$$a = e * (e * a). \tag{14}$$

Similarly to (7) the equivalence is valid:

$$a = b \text{ iff } e * a = e * b. \tag{15}$$

It is obvious in one of the directions. Let $e * a = e * b = p$. Then form (14):

$a * b = (e * (e * a)) * (e * (e * b)) = (e * p) * (e * p) = e.$

A groupide for which for every a, b, $c \in G$ the equation is valid:

$$(a * b) * c = a * (c * (e * b)) \tag{16}$$

will be called a R-groupide. Some examples of R-groupides are R , $R{+}_1$, $L_{0,1}$ and $L_{1,0}$. Similarly to (5) we shall define for every $a, b, c \in G$:

$$a * b = c \text{ iff } a = b \# c, \tag{17}$$

where # is a fixed binary operation (which may eventually coincide with @) defined over G. For that operation, obviously is valid:

$$a \# e = a. \tag{18}$$

□

Theorem 2: If (17) is valid and $\langle G, e, * \rangle$ is R-groupide, then $\langle G, e, \# \rangle$ is a right group.

Proof. It can be seen easily that (7) is valid again, and instead of (8) and (9) are valid:

$$b \# (a * b) = a, \tag{19}$$
$$(a \# b) * a = b. \tag{20}$$

We shall check that for every $a, b, c \in G$:

$$a \# b = b * (e * a). \tag{21}$$

Let $a \# b = x$. Hence $x * a = b$ and

$$b * (e * a) = (x * a) * (e * a) = x * ((e * a) * (e * a)) = x * e$$

i.e. (21) is present. We shall prove that for every $a, b, c \in G$

$$(e \# b) \# c = a \# (b \# c). \tag{22}$$

Let $(a * b) \# c = x$. Hence $c = x * (a \# b)$. If $y = a \# (b * c)$, then $y * a = b \# c$. i.e. $(y * a) * b = c$. Then from (21): $e = c * ((y * a) * b) = c * (y * (b * (e * a))) * c * (y * (a \# b))$, i.e. $c = y * (a \# b)$. Hence $* (a \# b) = y * (a \# b)$ and from (19)

$$x = (a \# b) \# (y * (a \# b)) = y,$$

i.e. (22) is present. We shall prove that-for every $a \in G$ there exists unique $b \in G$ for which:

$$a \# b = e.$$

Let $b = e * a$. From (19) it follows that $a \# (e * a) = e$. Let for a fixed $a \in G : a \# b = e$ and $a \# c = e$. Then:

$$b * c = (e * a) * (e * a) = e.$$

and from (7) it follows that the inverse element of G is unique. Hence $\langle G, e, \# \rangle$ is a right group. As in the first theorem, also here the opposite statement is not always valid. □

The groupide $\langle G, e, * \rangle$ will be called SL-groupide iff it is S-groupide and L-groupide.

Theorem 3: If $\langle G, e, * \rangle$ is SL-groupide, it is R-groupide.

Proof. Let $\langle G, e, * \rangle$ is SL-groupide. From (3) it follows directly the validity of (14). We shall prove that for every $a, b \in G$:

$$a * (a * b) = b * (e * a) \tag{23}$$

Indeed form (3), (4) and (14) we derive that

$$a * (e * b) = e * ((e * b) * a) = e * (e * (b * (e * a))) = b * (e * a)$$

The Eq. (16) is also valid because from (4) and (21):

$$(a * b) * c = a * (b * (e * cc)) = a * (c * (e * b)).$$

Hence $\langle G, e, * \rangle$ is R-groupide. □

We can define the object *SR*-groupide in the same way as we have already done above. The Eqs. (3) and (16) will be valid for it simultaneously and also if $\langle G, e, *\rangle$ is *SR*-groupide, then it is *L*-groupide.

We shall call a *LR*-groupide the groupide which is a *L*- and *R*-groupide. For it the equality (14) be valid, and hence:

$$a * b = (e * (e * a)) * b = e * (b * (e * (e * a))) = e * (b * a)$$

i.e., this groupide is *S*-groupide.

If we call *SLR*-groupide the groupide for which (3), (14) and (16) are valid we see that the objects *SL*-, *LR*-, *SR*- and *SLR*-groupide coincide.

One can easily check the validity of the following:

Theorem 4: If $\langle G, e, *\rangle$ is a *SLR*-groupide and (5) and (17) are present for the coinciding operations @ and # , then $\langle G, e, @\rangle$ is a commutative group.

Theorem 5: If S, L and T are respectively the sets of all *S*-, *L*- and *R*-groupide, then:

$$S \cap (L \cup T) = L \cap T.$$

Proof. From the fact that every *LR*-groupide is a *S*-groupide follows that:

$$L \cap T \subset S \cap (L \cup T).$$

On the other hand, as already is shown $S \cap L \subset T$ and $S \cap T \subset L$, i.e.,

$$(S \cap L) \cup (S \cap T) \subset T \cup (S \cap T) = T$$

and

$$(S \cap L) \cup (S \cap T) \subset L \cup (S \cap L) = L.$$

Hence,

$$(S \cap L) \cup (S \cap T) = S \cap (L \cup T) \subset T \cap L. \qquad \square$$

Finally we shall note that the above results generalize some results from [2, 3].

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1. Clifford A. H., G. B. Preston. (1964 & 1967). The Algebraic Theory of Semi-groups. Parts I and II, American Math Soc, Providence, Rhode Island.
2. Whittaker J. (1959). On the Structure of Half-groups, Canadian J of Math, 11(4), 651-659.
3. Radmanabhan R. (1969). Inverse Loops as Groupoids with One Law, J of London Math Soc, 1(2), 203-206.

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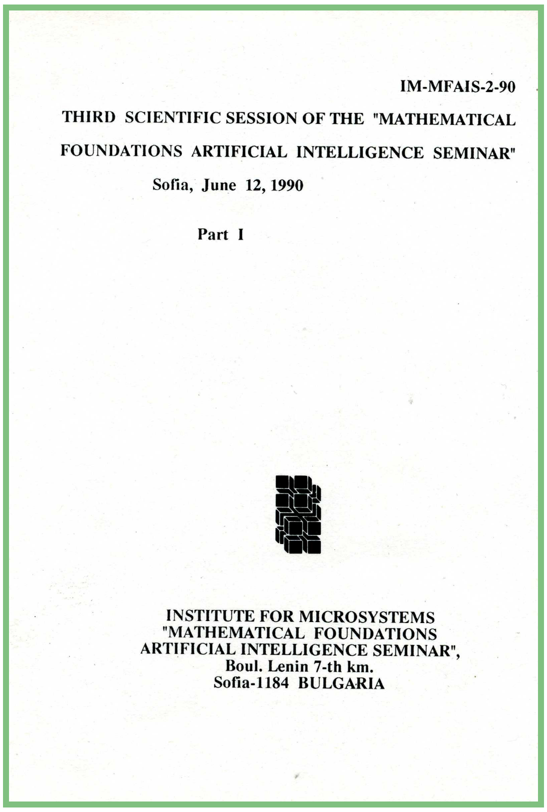

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THIRD SCIENTIFIC SESSION OF THE "MATHEMATICAL FOUNDATIONS ARTIFICIAL INTELLIGENCE SEMINAR"

Sofia, June 12, 1990

Part I

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"MATHEMATICAL FOUNDATIONS
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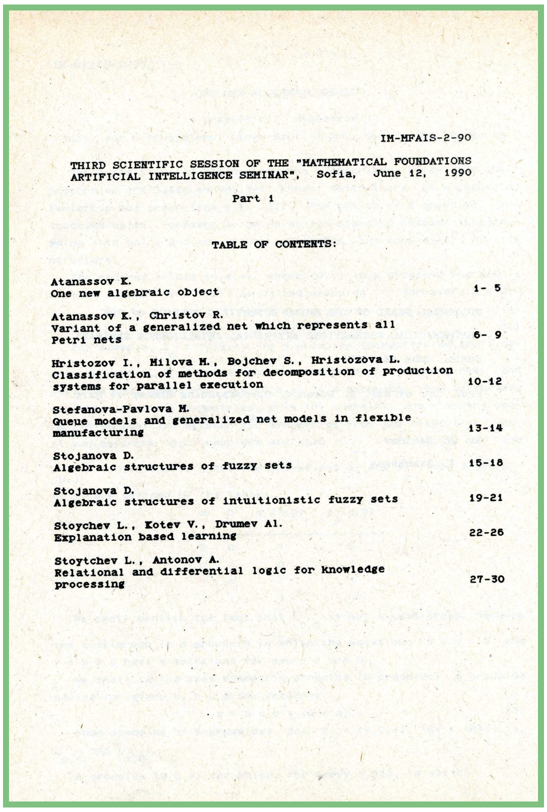
Cover page



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THIRD SCIENTIFIC SESSION OF THE "MATHEMATICAL FOUNDATIONS ARTIFICIAL INTELLIGENCE SEMINAR", Sofia, June 12, 1990

Part 1

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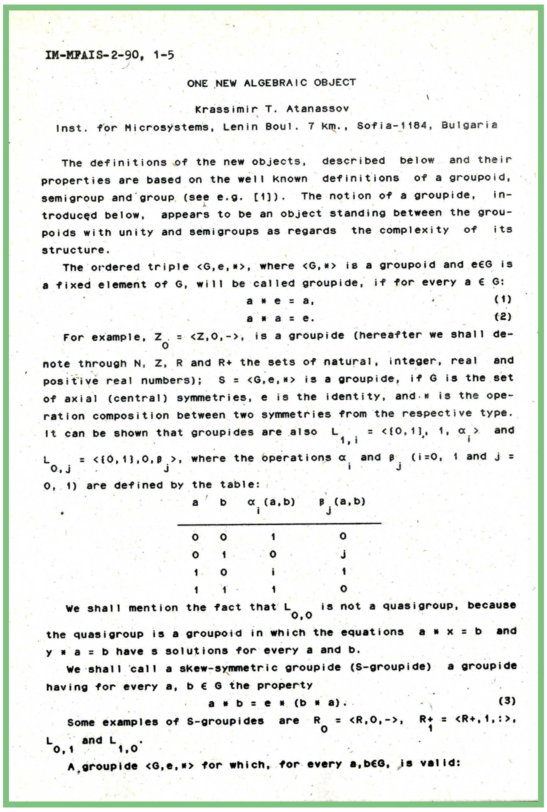

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ONE NEW ALGEBRAIC OBJECT

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The definitions of the new objects, described below and their properties are based on the well known definitions of a groupoid, semigroup and group (see e.g. [1]). The notion of a groupide, introduced below, appears to be an object standing between the groupoids with unity and semigroups as regards the complexity of its structure.

The ordered triple $\langle G,e,*\rangle$, where $\langle G,*\rangle$ is a groupoid and $e \in G$ is a fixed element of G, will be called groupide, if for every $a \in G$:

$$a * e = a, \qquad (1)$$

$$a * a = e. \qquad (2)$$

For example, $Z_0 = \langle Z,0,-\rangle$, is a groupide (hereafter we shall denote through N, Z, R and R+ the sets of natural, integer, real and positive real numbers); $S = \langle G,e,*\rangle$ is a groupide, if G is the set of axial (central) symmetries, e is the identity, and $*$ is the operation composition between two symmetries from the respective type. It can be shown that groupides are also $L_{1,i} = \langle\{0,1\}, 1, \alpha_i\rangle$ and $L_{0,j} = \langle\{0,1\},0,\beta_j\rangle$, where the operations α_i and β_j ($i=0$, 1 and $j = 0$, 1) are defined by the table:

a	b	$\alpha_i(a,b)$	$\beta_j(a,b)$
0	0	1	0
0	1	0	j
1	0	i	1
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We shall mention the fact that $L_{0,0}$ is not a quasigroup, because the quasigroup is a groupoid in which the equations $a * x = b$ and $y * a = b$ have s solutions for every a and b.

We shall call a skew-symmetric groupide (S-groupide) a groupide having for every $a, b \in G$ the property

$$a * b = e * (b * a). \qquad (3)$$

Some examples of S-groupides are $R_0 = \langle R,0,-\rangle$, $R+_1 = \langle R+,1,:\rangle$, $L_{0,1}$ and $L_{1,0}$.

A groupide $\langle G,e,*\rangle$ for which, for every $a,b \in G$, is valid:

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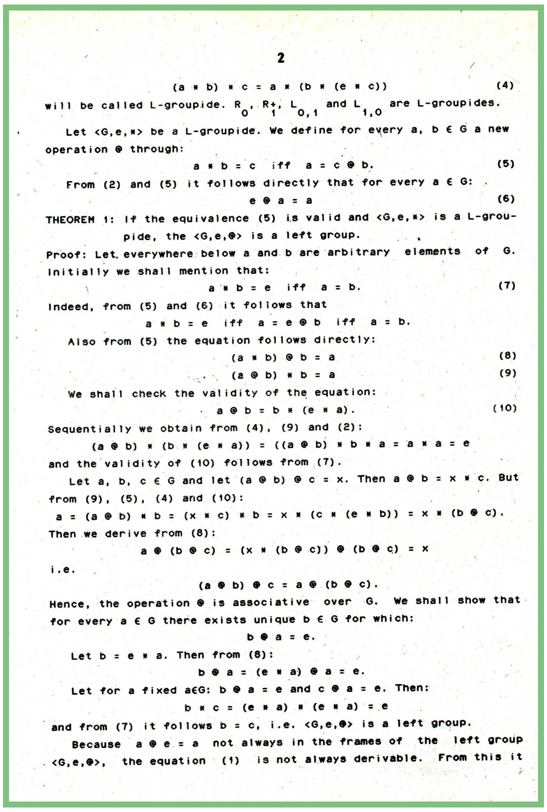

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$$(a * b) * c = a * (b * (e * c)) \qquad (4)$$

will be called L-groupide. R_0, $R+_1$, $L_{0,1}$ and $L_{1,0}$ are L-groupides.

Let $\langle G,e,*\rangle$ be a L-groupide. We define for every $a, b \in G$ a new operation @ through:

$$a * b = c \ \text{ iff } \ a = c @ b. \qquad (5)$$

From (2) and (5) it follows directly that for every $a \in G$:

$$e @ a = a \qquad (6)$$

THEOREM 1: If the equivalence (5) is valid and $\langle G,e,*\rangle$ is a L-groupide, the $\langle G,e,@\rangle$ is a left group.

Proof: Let everywhere below a and b are arbitrary elements of G. Initially we shall mention that:

$$a * b = e \ \text{ iff } \ a = b. \qquad (7)$$

Indeed, from (5) and (6) it follows that

$$a * b = e \ \text{ iff } \ a = e @ b \ \text{ iff } \ a = b.$$

Also from (5) the equation follows directly:

$$(a * b) @ b = a \qquad (8)$$

$$(a @ b) * b = a \qquad (9)$$

We shall check the validity of the equation:

$$a @ b = b * (e * a). \qquad (10)$$

Sequentially we obtain from (4), (9) and (2):

$$(a @ b) * (b * (e * a)) = ((a @ b) * b * a = a * a = e$$

and the validity of (10) follows from (7).

Let $a, b, c \in G$ and let $(a @ b) @ c = x$. Then $a @ b = x * c$. But from (9), (5), (4) and (10):

$$a = (a @ b) * b = (x * c) * b = x * (c * (e * b)) = x * (b @ c).$$

Then we derive from (8):

$$a @ (b @ c) = (x * (b @ c)) @ (b @ c) = x$$

i.e.

$$(a @ b) @ c = a @ (b @ c).$$

Hence, the operation @ is associative over G. We shall show that for every $a \in G$ there exists unique $b \in G$ for which:

$$b @ a = e.$$

Let $b = e * a$. Then from (8):

$$b @ a = (e * a) @ a = e.$$

Let for a fixed $a \in G$: $b @ a = e$ and $c @ a = e$. Then:

$$b * c = (e * a) * (e * a) = e$$

and from (7) it follows $b = c$, i.e. $\langle G,e,@\rangle$ is a left group.

Because $a @ e = a$ not always in the frames of the left group $\langle G,e,@\rangle$, the equation (1) is not always derivable. From this it

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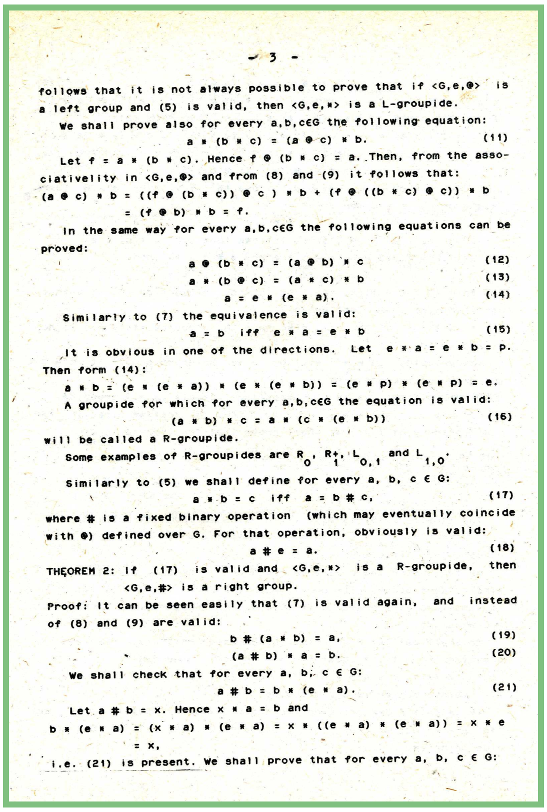

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follows that it is not always possible to prove that if $\langle G,e,@\rangle$ is a left group and (5) is valid, then $\langle G,e,*\rangle$ is a L-groupide.

We shall prove also for every $a,b,c\in G$ the following equation:

$$a * (b * c) = (a @ c) * b. \tag{11}$$

Let $f = a * (b * c)$. Hence $f @ (b * c) = a$. Then, from the associativelity in $\langle G,e,@\rangle$ and from (8) and (9) it follows that:

$$(a @ c) * b = ((f @ (b * c)) @ c) * b + (f @ ((b * c) @ c)) * b$$
$$= (f @ b) * b = f.$$

In the same way for every $a,b,c\in G$ the following equations can be proved:

$$a @ (b * c) = (a @ b) * c \tag{12}$$
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$$a = e * (e * a). \tag{14}$$

Similarly to (7) the equivalence is valid:

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It is obvious in one of the directions. Let $e * a = e * b = p$. Then form (14):

$$a * b = (e * (e * a)) * (e * (e * b)) = (e * p) * (e * p) = e.$$

A groupide for which for every $a,b,c\in G$ the equation is valid:

$$(a * b) * c = a * (c * (e * b)) \tag{16}$$

will be called a R-groupide.

Some examples of R-groupides are R_0, $R+_1$, $L_{0,1}$ and $L_{1,0}$.

Similarly to (5) we shall define for every $a, b, c \in G$:

$$a * b = c \text{ iff } a = b \# c, \tag{17}$$

where # is a fixed binary operation (which may eventually coincide with @) defined over G. For that operation, obviously is valid:

$$a \# e = a. \tag{18}$$

THEOREM 2: If (17) is valid and $\langle G,e,*\rangle$ is a R-groupide, then $\langle G,e,\#\rangle$ is a right group.

Proof: It can be seen easily that (7) is valid again, and instead of (8) and (9) are valid:

$$b \# (a * b) = a, \tag{19}$$
$$(a \# b) * a = b. \tag{20}$$

We shall check that for every $a, b, c \in G$:

$$a \# b = b * (e * a). \tag{21}$$

Let $a \# b = x$. Hence $x * a = b$ and

$$b * (e * a) = (x * a) * (e * a) = x * ((e * a) * (e * a)) = x * e$$
$$= x,$$

i.e. (21) is present. We shall prove that for every $a, b, c \in G$:

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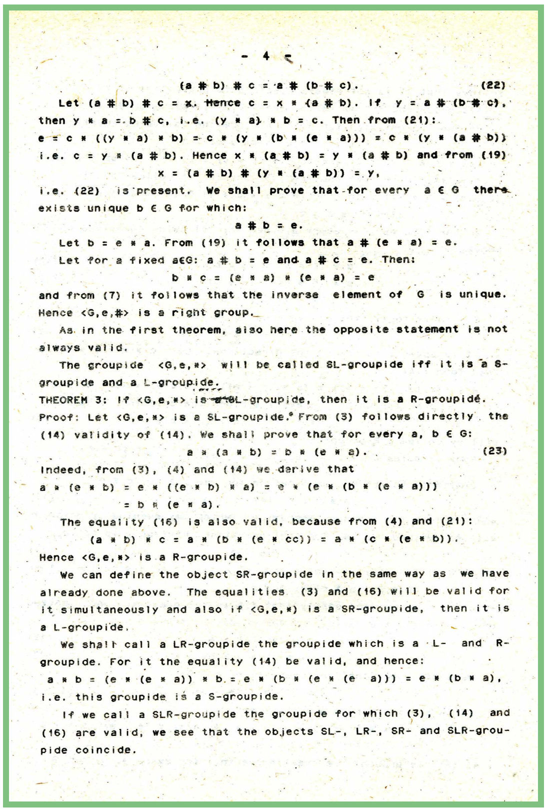

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$$(a \# b) \# c = a \# (b \# c). \tag{22}$$

Let $(a \# b) \# c = x$. Hence $c = x * (a \# b)$. If $y = a \# (b \# c)$, then $y * a = b \# c$, i.e. $(y * a) * b = c$. Then from (21):

$$e = c * ((y * a) * b) = c * (y * (b * (e * a))) = c * (y * (a \# b))$$

i.e. $c = y * (a \# b)$. Hence $x * (a \# b) = y * (a \# b)$ and from (19)

$$x = (a \# b) \# (y * (a \# b)) = y,$$

i.e. (22) is present. We shall prove that for every $a \in G$ there exists unique $b \in G$ for which:

$$a \# b = e.$$

Let $b = e * a$. From (19) it follows that $a \# (e * a) = e$.

Let for a fixed $a\in G$: $a \# b = e$ and $a \# c = e$. Then:

$$b * c = (e * a) * (e * a) = e$$

and from (7) it follows that the inverse element of G is unique. Hence $\langle G,e,\#\rangle$ is a right group.

As in the first theorem, also here the opposite statement is not always valid.

The groupide $\langle G,e,*\rangle$ will be called SL-groupide iff it is a S-groupide and a L-groupide.

THEOREM 3: If $\langle G,e,*\rangle$ is a SL-groupide, then it is a R-groupidé.

Proof: Let $\langle G,e,*\rangle$ is a SL-groupide. From (3) follows directly the (14) validity of (14). We shall prove that for every $a, b \in G$:

$$a * (a * b) = b * (e * a). \tag{23}$$

Indeed, from (3), (4) and (14) we derive that

$$a * (e * b) = e * ((e * b) * a) = e * (e * (b * (e * a)))$$
$$= b * (e * a).$$

The equality (16) is also valid, because from (4) and (21):

$$(a * b) * c = a * (b * (e * cc)) = a * (c * (e * b)).$$

Hence $\langle G,e,*\rangle$ is a R-groupide.

We can define the object SR-groupide in the same way as we have already done above. The equalities (3) and (16) will be valid for it simultaneously and also if $\langle G,e,*)$ is a SR-groupide, then it is a L-groupide.

We shall call a LR-groupide the groupide which is a L- and R-groupide. For it the equality (14) be valid, and hence:

$$a * b = (e * (e * a)) * b = e * (b * (e * (e \;\; a))) = e * (b * a),$$

i.e. this groupide is a S-groupide.

If we call a SLR-groupide the groupide for which (3), (14) and (16) are valid, we see that the objects SL-, LR-, SR- and SLR-groupide coincide.

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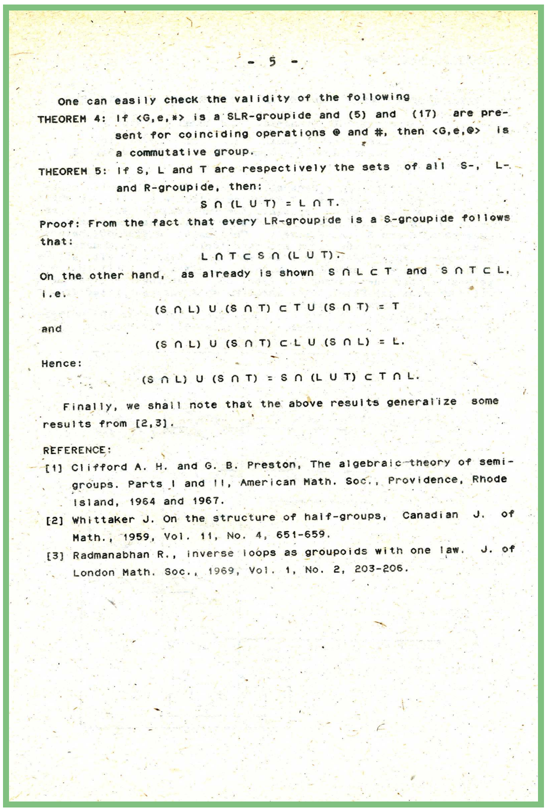

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One can easily check the validity of the following

THEOREM 4: If $\langle G,e,*\rangle$ is a SLR-groupide and (5) and (17) are present for coinciding operations @ and #, then $\langle G,e,@\rangle$ is a commutative group.

THEOREM 5: If S, L and T are respectively the sets of all S-, L- and R-groupide, then:

$$S \cap (L \cup T) = L \cap T.$$

Proof: From the fact that every LR-groupide is a S-groupide follows that:

$$L \cap T \subset S \cap (L \cup T).$$

On the other hand, as already is shown $S \cap L \subset T$ and $S \cap T \subset L$, i.e.

$$(S \cap L) \cup (S \cap T) \subset T \cup (S \cap T) = T$$

and

$$(S \cap L) \cup (S \cap T) \subset L \cup (S \cap L) = L.$$

Hence:

$$(S \cap L) \cup (S \cap T) = S \cap (L \cup T) \subset T \cap L.$$

Finally, we shall note that the above results generalize some results from [2,3].

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