A Numerical Solution of Volterra's Population Growth Model Based on Hybrid Function

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Received: August 08, 2016 **Accepted: February 16, 2017**

Published: March 31, 2017

Abstract: *In this paper, a new numerical method for solving Volterra's population growth model is presented. Volterra's population growth model is a nonlinear integro-differential equation. In this method, by introducing the combination of fourth kind of Chebyshev polynomials and Block-pulse functions, approximate solution is presented. To do this, at first the interval of equation is divided into small sub-intervals, then approximate solution is obtained for each sub-interval. In each sub-interval, approximate solution is assumed based on introduced combination function with unknown coefficients. In order to calculate unknown coefficients, we imply collocation method with Gauss-Chebyshev points. Finally, the solution of equation is obtained as the sum of solutions at all sub-intervals. Also, it has been shown that upper bound error of approximate solution is* $O\left(\frac{m^{-r}}{\sqrt{N}}\right)$. *It means that by increasing m and N, error will decrease. At the end, the comparison of numerical results with some existing ones, shows high accuracy of this method.*

Keywords: *Integro-differential equation, Chebyshev polynomials, Block-pulse functions, Gauss-Chebyshev points, Hybrid function.*

Introduction

Nowadays, much attention has been pointed to integral equations. This is because the most practical problems in science lead to these equations. One of the problems existing in the population growth study is Volterra equation of population growth. The following equation has been introduced by Volterra for population growth model:

$$\frac{dp}{d\hat{t}} = ap - bp^2 - cp\int_0^{\hat{t}} p(x)dx, \quad p(0) = p_0, \tag{1}$$

where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, $c > 0$ is the toxicity factor, p_0 is the initial population and $p(\hat{t})$ is the population at time $\hat{t}$.

Likewise, $cp\int_0^{\hat{t}} p(x)dx$ includes the accumulated toxicity when the time goes to zero, respectively [17, 18].

However, using the following variables:

$$t = \frac{\hat{t}}{\frac{b}{c}}, \quad u = \frac{p}{\frac{a}{b}},$$

the following equation is obtained from Eq. (1):

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, \qquad u(0) = u_0, \tag{2}$$

where $\kappa = \frac{c}{ab}$ is predictive dimensionless parameter and $u(t)$ is the population at t. As far as the environment permits, the systems population tends to increase. The majority of the population undergoes a dynamic process to reach a balance point. The number of population increases in a sensitive balance point which is the result of some limited factors. These factors include:

- Nutritional components;
- Crowding;
- Competition;
- Increasing the concentration of waste.

For more details see [19].

In addition, if the event of sudden deaths such as deaths arising from earthquakes, it is called the collapse of the population. Given that Eq. (2) has no analytical solutions, numerical methods for solving it are highly regarded. Over the past two decades, several methods for the numerical solution of this equation have been presented. To solve this equation, Euler and modified Euler, Fourth order Runge-Kutta and Fehelberg Runge-Kutta methods were provided in 1997 [17]. These methods have high computational bulk to calculate the answer in one point. Another approach based on Adomian decomposition method to solve the equation was presented by Wazwaz [18]. Although the efficiency of these methods is simply further by increasing the number of sentences series but both computational complexity and rounding error increase. Also Adomian decomposition method compared with Sinc Galerkin method and showed the Adomian decomposition method is more efficient and accurate in solving this equation [1]. The methods of singular perturbation [16], spectral [12–14] and radial basis functions [10] used to solve this equation and also their sensitivity of growth for different values of κ, have been examined. Hybrid functions used to solve this equation [6, 8]. The hybrid functions (Block-pulse functions and Legendre polynomials) used to solve this equation [8]. Rational pseudospectral method was proposed by Dehghan and co-workers in 2015 [2]. Kajani et al. also introduced the multi-domain pseudospectral method to solve population growth equation [7]. By comparing the numerical result of the present method with some above mentioned methods, accuracy and efficiency of the proposed method are shown.

In the following, at first the Block-pulse functions and fourth kind of Chebyshev polynomials are presented and a combination of them is introduced. In Section 5 approximate solution to the equation is provided by the combined functions. By substituting the combined function into the Eq. (2), a system of nonlinear equations is achieved. In Section 6, an upper bound for error of the approximate solution has been obtained. Finally, Section 7 shows numerical results and a comparison with other methods.

Fourth kind of Chebyshev polynomials

A Sturm-Liouville problem is an eigenvalue problem on the interval $(-1,1)$ as follows:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u(x)=\lambda w(x)u(x). \tag{3}$$

By assuming $p(x)=(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}$, $q(x)=0$ and $w(x)=(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$ fourth kinds of Chebyshev polynomials are obtained as follows:

$$w_n(x)=\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \tag{4}$$

where $x=\cos\theta$. On the basis of Eq. (4) recurrence relations to them are:

$$\begin{cases} w_n(x)=2xw_{n-1}(x)-w_{n-2}(x), & n=2,3,\ldots, \\ w_1(x)=2x+1, \quad w_0(x)=1. \end{cases} \tag{5}$$

This polynomials are particular type of Jacobi polynomials per $\alpha=1/2$ and $\beta=-1/2$:

$$\binom{2n}{n}w_n(x)=2^{2n}J_n^{(\frac{1}{2},-\frac{1}{2})}(x). \tag{6}$$

Special cases of Jacobi polynomials have been used in many numerical methods [3, 11, 19].

The polynomials on the interval $(-1,1)$ relative to the weight function

$$w(x)=(1-x)^{\frac{1}{2}}(1+x)^{\frac{-1}{2}}$$

are orthogonal and:

$$\int_{-1}^{1}w_i(x)w_j(x)w(x)dx=\pi\delta_{ij}, \tag{7}$$

where δ_{ij} is the Kronecker function [9]. The roots of fourth type of Chebyshev polynomials of degree m are shown with γ_i, so:

$$-1<\gamma_1<\gamma_2<\cdots<\gamma_m<1, \tag{8}$$

and γ_i $(i=1,2,\ldots,m)$ are called fourth kind Gauss-Chebyshev points.
In this method Gauss-Chebyshev points will be used as collocation points subsequently.

Integral of fourth kind of Chebyshev polynomials

To solve population growth equation by the present method, the integrals of these polynomials are needed which are obtained from the following equations:

If $i=2k$,

$$\begin{aligned} \int_{-1}^{x}w_{2i}(t)dt &= \frac{-1}{2i}w_0(x)-\frac{1}{4i}w_{2i-2}(x)+\frac{1}{4i(2i+1)}w_{2i}(x)+\frac{1}{2(2i+1)}w_{2i+1}(x), \\ \int_{-1}^{x}w_{2i-1}(t)dt &= \frac{-1}{2i}w_0(x)-\frac{1}{2(2i-1)}w_{2i-2}(x)+\frac{1}{4i(2i-1)}w_{2i-1}(x)+\frac{1}{4i}w_{2i}(x) \end{aligned} \tag{9}$$

and if $i = 2k-1$,

$$\begin{aligned}\int_{-1}^{x} w_{2i}(t)dt &= \frac{1}{2i}w_0(x) - \frac{1}{4i}w_{2i-1}(x) + \frac{1}{4i(2i+1)}w_{2i}(x) + \frac{1}{2(2i+1)}w_{2i+1}(x),\\ \int_{-1}^{x} w_{2i-1}(t)dt &= \frac{1}{2i}w_0(x) - \frac{1}{2(2i-1)}w_{2i-2}(x) + \frac{1}{4i(2i-1)}w_{2i-1}(x) + \frac{1}{4i}w_{2i}(x)\end{aligned} \tag{10}$$

and for $n = 0, 1$:

$$\begin{aligned}\int_{-1}^{x} w_0(t)dt &= \frac{1}{2}w_0(x) + \frac{1}{2}w_1(x),\\ \int_{-1}^{x} w_1(t)dt &= \frac{1}{4}w_1(x) + \frac{1}{4}w_2(x).\end{aligned} \tag{11}$$

Proof. The Proof is clear and hence omitted. □

Block-pulse functions

Block-pulse functions with $b_i(\lambda)$, $i = 1, \ldots, N$ on the interval $[0, T)$ are shown as follows:

$$b_i(\lambda) = \begin{cases} 1, & \frac{(i-1)T}{N} \leq \lambda < \frac{iT}{N},\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

These functions have three properties:

- disjointness;
- orthogonality;
- completeness.

These properties make use of these functions, easy operations and produce satisfactory approximations [5].

Hybrid functions

Hybrid functions of fourth kind of Chebyshev and block-pulse function with $h_{i,j}(t)$ as shown for $i = 1, \ldots, N$, $j = 0, \ldots, M-1$ are as follows:

$$h_{i,j}(t) = \begin{cases} \sqrt{\frac{2T}{N}}w_j(\frac{2N}{T}t - 2i + 1), & \frac{(i-1)T}{N} \leq t < \frac{iT}{N},\\ 0, & \text{otherwise.}\end{cases} \tag{13}$$

Likewise, integral of $h_{i,j}(t)$ for $x \in \left[\frac{(i-1)T}{N}, \frac{iT}{N}\right]$ from the equation above is obtained as follows:

$$\int_{-1}^{x} h_{i,j}(t)_{dt} = \begin{cases} \sqrt{\frac{2T}{N}}\int_{-1}^{x} w_j(\frac{2N}{T}t - 2i + 1)dt, & \frac{(i-1)T}{N} \leq t < \frac{iT}{N},\\ 0, & \text{otherwise.}\end{cases} \tag{14}$$

Eqs. (13) and (14) can be presented in terms of Chebyshev polynomials via Eqs. (9)-(11).

Approximation function

To solve the Volterra integro-differential equation of population growth Eq. (2) on time interval $[0, T]$, at first the interval is divided into N sub-interval as $I_i = \left[\frac{(i-1)T}{N}, \frac{iT}{N}\right]$, $i = 1, \ldots, N$ then the approximation solution in each sub-interval is calculated. Finally, the sum of provided solutions in each sub-interval is presented as approximate solution of equation. For this reason, approximate solution of equation on i-th sub-interval is shown by $\hat{u}_i(x)$. At first, the derivative of approximate solution is approximated as follows:

$$\left.\frac{du(x)}{dx}\right|_{I_i} \simeq \frac{d\hat{u}_i(x)}{dx} = \sum_{j=0}^{m-1} h_{i,j}(x)c_{ij}, \tag{15}$$

where $h_{i,j}$ is introduced as hybrid function in Eq. (13) and $c_{i,j}$ is unknown coefficient. Needless to stay, the approximate solution $\hat{u}_{i,j}$ is obtained by calculating the integral of the above equation on interval $\left[\frac{(i-1)T}{N}, x\right]$ shown in the following equation:

$$u(x)|_{I_i} \simeq \hat{u}_i(x) = \sum_{j=0}^{m-1} \left(\int_{\frac{(i-1)T}{N}}^{x} h_{i,j}(t)dtc_{ij} \right) + \hat{u}_i\left(\frac{(i-1)T}{N}\right), \tag{16}$$

where

$$\hat{u}_i\left(\frac{(i-1)T}{N}\right) = \begin{cases} u_0, & i = 1, \\ \hat{u}_{i-1}\left(\frac{(i-1)T}{N}\right), & i = 2, \ldots, N \end{cases} \tag{17}$$

and u_0 is the initial condition from Eq. (2). Also, integral of approximation function of Eq. (16) is obtained as follows:

$$\int_0^x \hat{u}_i(t)dt = \int_0^x \sum_{j=0}^{m-1} \left(\int_{\frac{(i-1)T}{N}}^{t} h_{i,j}(s)ds \right) c_{ij}dt + x\hat{u}_i\left(\frac{(i-1)T}{N}\right). \tag{18}$$

With an approximation function $\hat{u}_{i,j}$ placed in Eq. (2) we have:

$$\kappa\frac{d\hat{u}_i(x)}{dx} \simeq \hat{u}_i(x) - (\hat{u}_i(x))^2 - \hat{u}_i(x)\int_0^x \hat{u}_i(t)dt, \quad x \in I_i. \tag{19}$$

By using the Eqs. (15)-(18) and collocation method with collocation point x_k^i we will have:

$$x_k^i = \frac{2N}{k}\left(\gamma_k - \frac{iT}{N}\right) + 1, \quad k = 1, \ldots, m,$$

where γ_k is Gauss-Chebyshev point introduced in Eq. (8), the $m \times m$ nonlinear system of algebraic equations is achieved as follows:

$$\kappa\frac{d\hat{u}(x_k^i)}{dx} - \hat{u}_i(x_k^i) + \left(\hat{u}_i\left(x_k^i\right)\right)^2 + \hat{u}_i(x_k^i)\int_0^{x_k^i} \hat{u}_i(t)dt = 0, \quad k = 1, \ldots, m. \tag{20}$$

By solving the above system with *fsolve* function of the Maple software, coefficients $c_{i,j}$ in i-th sub-interval are calculated. By applying this procedure on all sub-intervals approximate solution is obtained in each sub-interval. Finally, approximate solution of Eq. (2) is obtained as follows:

$$\hat{u}(x) \simeq \sum_{i=1}^{N} \hat{u}_i(x). \tag{21}$$

Error analysis

In this section, the upper bound of approximation function will be obtained.

Theorem. Suppose that $u \in H^r_{\chi^{(\alpha,\beta)},A}(A)$ (r is a non-negative integer) $\alpha = 1/2$ and $\beta = -1/2$ then:

$$\|L_m^{(-1,1)}u - u\|_{L^2} \leq cm^{-r}\left(\int_{-1}^{1}(1-t)^{r+\frac{1}{2}}(1+t)^{r-\frac{1}{2}}\left(\frac{d^r u(t)}{dt^r}\right)^2 dt\right)^{\frac{1}{2}}, \tag{22}$$

where $L_m^{(-1,1)}u = \hat{u}(t)$, $c(\alpha,\beta)$ is a constant dependent on α, β and $H^r_{\chi^{(\alpha,\beta)},A}(A)$ is the weighted Sobolev space on the interval A with weight function $\chi^{(\alpha,\beta)}$.

Proof. By considering Eq. (6) fourth kinds of Chebyshev polynomials are obtained from Jacobi polynomials. Therefore, by assuming $\alpha = 1/2$, $\beta = -1/2$ and theorem (4.3) of [4], we have:

$$\|L_{G,m,\alpha,\beta}u - u\|_{\chi^{(\gamma,\delta)}} \leq c_{\alpha,\beta}(m(m+\alpha+\beta))^{-\frac{r}{2}}|u|_{r,\chi^{(\alpha,\beta)},A}, \tag{23}$$

where m is the degree of polynomial and $L_{G,m,\alpha,\beta}u$ provides approximation function of Jacobi polynomials. And semi-norm in Eq. (23) is as follows:

$$|u|_{r,\chi^{(\alpha,\beta)},A} = \|\partial_t^r u\|_{\chi^{(\alpha+r,\beta+r)}} = \left(\int_A x^{(\alpha+r,\beta+r)}\left(\frac{\partial^r u}{\partial t}\right)^2 dt\right)^{\frac{1}{2}} \tag{24}$$

and $\chi^{(\alpha+r,\beta+r)}$ is the weight function in this method as follows:

$$\chi^{(\frac{1}{2}+r,-\frac{1}{2}+r)} = (1+t)^{r-\frac{1}{2}}(1-t)^{r+\frac{1}{2}},$$

by substituting Eq. (24) into Eq. (23) and using $\alpha = \gamma = 1/2$ and $\beta = \delta = -1/2$, Eq. (22) will be obtained. □

It can be stated that upper bound of the error of the approximation function is $o\left(\frac{m^{-r}}{\sqrt{N}}\right)$. This means that error decreases by increasing m. On the other hand by considering the algorithm in this way the upper bound of the error in the i-th sub-interval $\left[\frac{(i-1)T}{N}, \frac{iT}{N}\right]$ according to Eq. (2) is obtained as follows:

$$\|\hat{u}_i(x) - u\| \leq c(\alpha,\beta)m^{-r}\left(\int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}}\left(t - \frac{(i-1)T}{N}\right)^{r+\frac{1}{2}}\left(\frac{iT}{N} - t\right)^{r-\frac{1}{2}}\left(\frac{d^r u(t)}{dt^r}\right)^2 dt\right)^{\frac{1}{2}}. \tag{25}$$

By using the mean value theorem for integral in Eq. (25), we get:

$$\int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} \left(t-\frac{(i-1)T}{N}\right)^{r+\frac{1}{2}} \left(\frac{iT}{N}-t\right)^{r-\frac{1}{2}} \left(\frac{d^r u(t)}{dt^r}\right)^2 dt \leq M_i \frac{T}{N},$$

where M_i is defined by the following equation:

$$M_i = \max_{\frac{(i-1)T}{N}\leq t\leq \frac{iT}{N}} \left(t-\frac{(i-1)T}{N}\right)^{r+\frac{1}{2}} \left(\frac{iT}{N}-t\right)^{r-\frac{1}{2}} \left(\frac{d^r u(t)}{dt^r}\right)^2,$$

then:

$$\|\hat{u}_i(x)-u\| \leq c(\alpha,\beta)m^{-r}\sqrt{\frac{M_iT}{N}}. \tag{26}$$

It means that, the upper bound of the error in this sub-interval is $o\left(\frac{m^{-r}}{\sqrt{N}}\right)$. Therefore, it can be said that by increasing m, N the error and accuracy will be decreased and increased, respectively. Obviously, this will be seen in the numerical results of the next section.

Numerical results

In this section integro-differential of population growth is solved by using the proposed method and the obtained results are compared with other methods. Efficiency and accuracy of this method are clearly specified by comparing numerical results with other methods. Consider the following population growth equation:

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx, \quad u(0) = 0.1\ . \tag{27}$$

This equation is solved for different values of κ. To solve this equation by present method, Maple 18 software and PC core-i7 2.4 GHZ are used. Note that the maximum value of u appeared with u_{max}, can be accurately calculated by the following equation [17]:

$$u_{\max} = 1 + \kappa \ln\left(\frac{\kappa}{1+\kappa-u_0}\right). \tag{28}$$

First, Eq. (27) with $\kappa = 0.02, 0.04, 0.1, 0.2, 0.5$ is solved.

The absolute error of u_{max} for different values of κ is presented in Table 1.

Table 1. Absolute error of $u_{\max}$ for different values of κ

κ	m	N	Absolute error of $u_{\max}$
0.02	15	30	3.28×10^{-8}
0.04	15	30	1.43×10^{-12}
0.1	15	30	2.23×10^{-18}
0.2	15	30	5.52×10^{-24}
0.5	15	30	6.17×10^{-27}

In Table 2 in terms of accuracy, the present method is compared with methods from [2, 6, 11, 15, 18].

Table 2. A comparison of absolute error of $u_{\max}$ obtained by the present method for $m = 15$, $N = 30$ with some other methods by different values of κ

κ	Present method	[6]	[18]	[2]	[15]	[11]
0.02	3.28×10^{-8}	4.30×10^{-3}	1.96×10^{-2}	3.72×10^{-7}	7.72×10^{-7}	6.95×10^{-6}
0.04	1.43×10^{-12}	4.56×10^{-3}	1.25×10^{-2}	1.43×10^{-8}	7.83×10^{-7}	4.15×10^{-5}
0.1	2.33×10^{-18}	5.27×10^{-3}	4.63×10^{-3}	1.07×10^{-10}	5.91×10^{-7}	3.93×10^{-8}
0.2	5.52×10^{-24}	3.06×10^{-3}	1.14×10^{-3}	3.53×10^{-11}	6.82×10^{-7}	8.16×10^{-6}
0.5	6.17×10^{-27}	2.49×10^{-3}	9.21×10^{-5}	2.44×10^{-9}	4.91×10^{-7}	1.19×10^{-7}

It can be seen from Table 2 that the method presented in this paper is more accurate than other methods. Tables 3 and 4 show that by increasing m and N the absolute error decreases as expected.

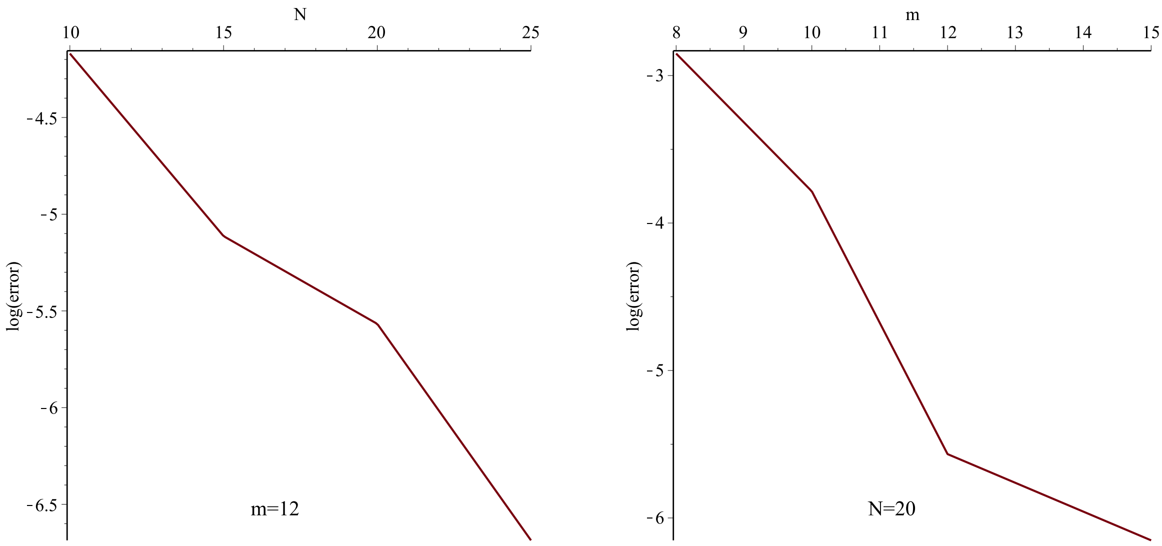
Table 3. Absolute error of $u_{\max}$ for $\kappa = 0.02$ by increasing m, N

κ	N	m	Absolute error of $u_{\max}$
0.02	20	8	1.41×10^{-3}
0.02	20	10	1.64×10^{-4}
0.02	20	12	2.23×10^{-6}
0.02	20	15	7.05×10^{-7}
0.02	10	12	6.77×10^{-5}
0.02	15	12	7.70×10^{-6}
0.02	20	12	2.71×10^{-6}
0.02	25	12	2.06×10^{-7}

Table 4. Absolute error of $u_{\max}$ for $\kappa = 0.5$ by increasing m, N

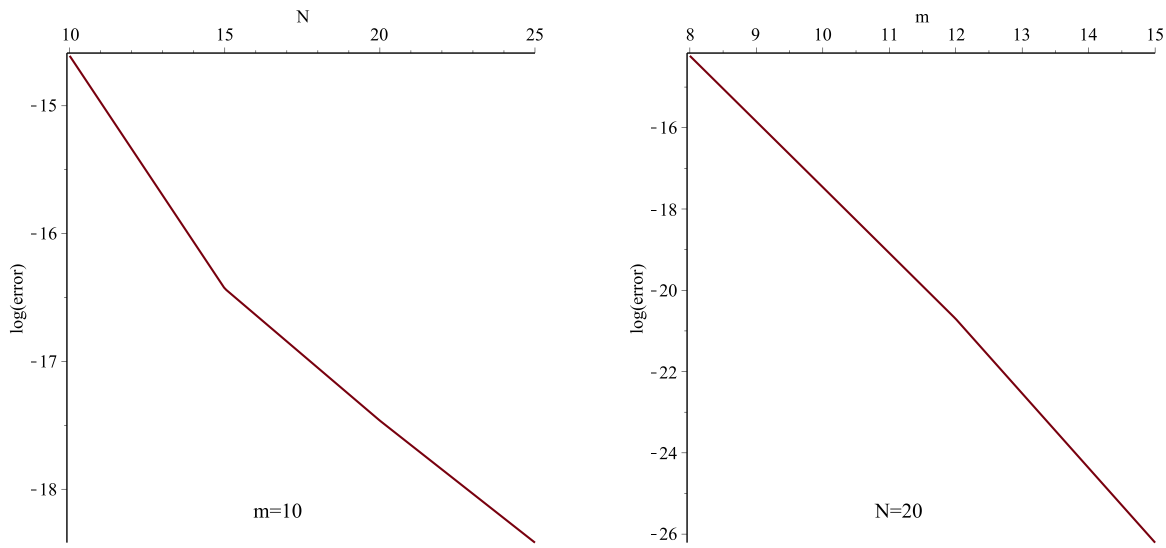
κ	N	m	Absolute error of $u_{\max}$
0.5	20	8	5.87×10^{-15}
0.5	20	10	3.44×10^{-18}
0.5	20	12	1.98×10^{-21}
0.5	20	15	6.17×10^{-27}
0.5	10	10	2.46×10^{-15}
0.5	15	10	3.71×10^{-17}
0.5	20	10	3.44×10^{-18}
0.5	25	10	3.82×10^{-19}

In addition Fig. 1 and Fig. 2 show that by increasing m, N the absolute error of u_{max} decreases.



(a) for $m = 12$ by increasing N (b) for $N = 20$ by increasing m

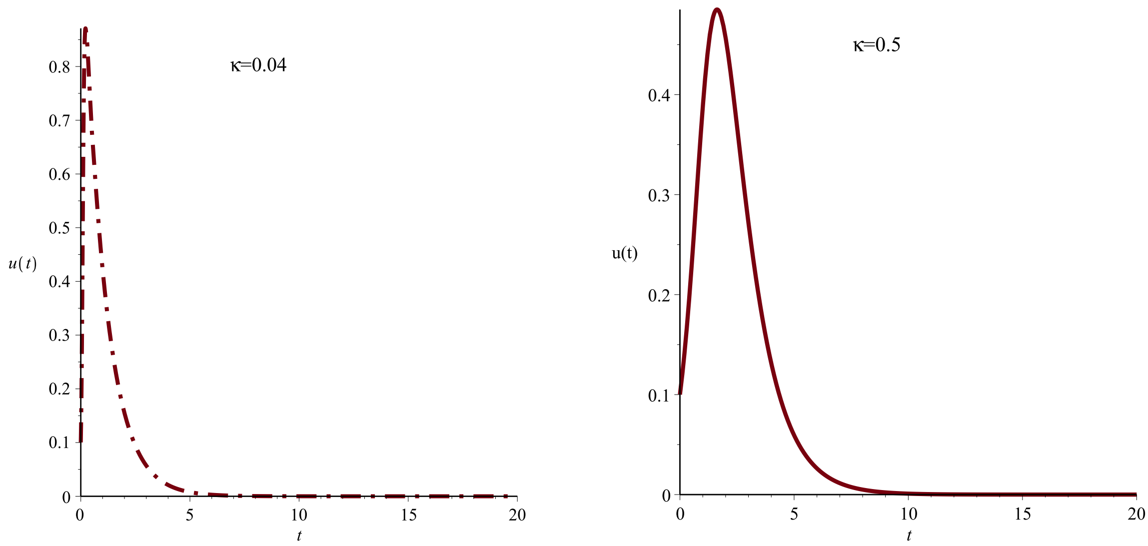
Fig. 1 The graph of absolute error of $u_{\max}$ for $\kappa = 0.02$



(a) for $m = 10$ by increasing N (b) for $N = 10$ by increasing m

Fig. 2 The graph of absolute error of $u_{\max}$ for $\kappa = 0.5$

Finally, in Fig. 3 the graph of approximate solution of $u(t)$ is presented. As noted above, one feature of this method is the ability to solve the equation on large domain, so the equation has been solved on $[0, 20]$ and its graph is shown in Fig. 3.



(a) for $\kappa = 0.04$ (b) for $\kappa = 0.5$

Fig. 3 The graph of approximate solution of $u(t)$

Conclusion

In this study, a new numerical method, hybrid function of the fourth kind Chebyshev polynomials and Block-Pulse functions was proposed to solve Volterra's population growth model. An important feature of this method is its high accuracy. Another advantage of our method is the capability of solving this equation on large domain.

Our scheme has been compared to several methods presented in the literature. The comparison of the results showed that the suggested method is more accurate than the other methods.

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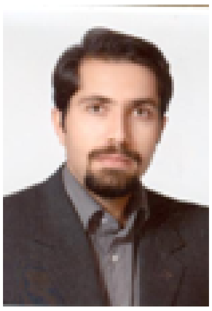

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