An Analysis of the "Horizontal Y" Shaped Queuing Model to Assist in Health Care Institution

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Abstract: *The "Horizontal Y" shaped queuing model in which there is bulk infinite arrival of patients, but available M services are limited. The discrete flow of patients in the system is reduced in continuous flow and a diffusion equation is used. In terms of means and variances of inter-arrival time distribution, this process of the number of patients and the number of servers is used, imposing reflecting boundaries. The inspiration for this analysis came from Aradhye and Kallurkar [2], and Armony [3], who explained that hospitals are complex systems whose problems can be resolved utilizing queuing theory. The present paper deals with a double-ended queue in which patients wait in a queue for services. The discrete distribution equations for queue size with various cases have been derived. Finally, the expected length of the queue, i.e., L_C and the expected finite server, i.e., L_S have been derived.*

Keywords: *Horizontal Y-shaped model, Health care institution, Model $G_\infty/G_M/1$, Bulk arrival.*

Introduction

Queuing theory is a broad discipline with roots dating back to the early twentieth century. The basic goal of queuing theory research is to figure out how to provide patients with high-quality and fast service. Erlang [5] did the groundwork on the subject of queuing theory. According to [2], waiting is a global problem that affects almost everyone, and it consumes a lot of time and money. Armony [3], stated that hospitals are extremely challenging systems with critical societal advantages. Many other researchers namely Lakshmi and Iyar [13], Pandey and Gangeshwer [15], Donatus et al. [4], Kalwar et al. [11], Tyagi et al. [22], Silva et al. [20], Shih [19], Raj et al. [16], and Li et al. [14] have worked on queuing theory. A physical perspective of patient flow in hospitals using a queuing model is useful for assessing and improving overall execution. The importance of quality and service work in health care is something that they should consider when planning and executing their operations or in the actual timetable. In terms of layout, capabilities, and control, queuing models are especially beneficial. Therefore, in this investigation, major attention has been given to the development of queuing models with their application in medical science.

Queuing models are extremely important for all of us because we frequently usually get encountered waiting lines or lineups. For example, a double-ended queue describes a situation in which a demand process is arriving at the queue, as well as a supply process that is arriving to match the demand process. The double-ended queue application settings range from organ transplants when patients are waiting for organs would be matched with donors or organs already obtained. In the traditional model, both the demand and supply processes are arrived in single units and demand will need only a single unit of service. At the time of arrival for both sides, if there is a queue of attendants waiting for the arrival of the unit under consideration, then, the demand and supply are matched instantaneously and leave the system together. It is also assumed that as long as a match demand unit and a supply unit come across each other, the match will happen without failure. The arrival processes were initially assumed to be exponential as well. A depiction of this system could also be seen in Fig. 1 below.

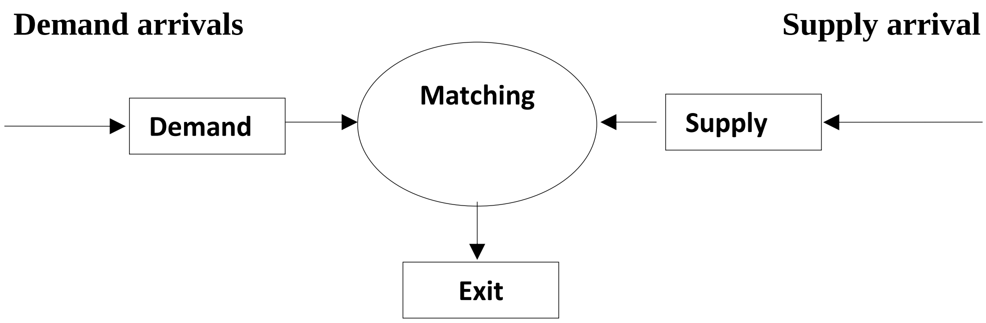

Fig. 1 Generalized double-ended queue system

Ahmed et al. [1] used single and multi-server exponential queuing systems, making the balance between excessive waiting time and cost in some selected Hospitals in North-Western Nigeria.

Using the generating function method, Sasieni [18] constructed a double-ended queue model with impatient clients. According to [17], diffusion approximation is an attempt to overcome the limits of exponential server queuing models by considering the mean and variance of the service time distribution. Whit [23] talked about how to improve queue diffusion approximations. In this case of the double-ended queue model, Srivastava and Kashyap [21] derived the generating function method and utilized it to analyze additional models. Hlynka and Sheahan [7] investigated the optimal management of one Poisson process over another. Due to mathematical complexity, the exact answer for a double-ended queue with a generic arrival distribution cannot be found. Diffusion approximation is used to solve complex queuing systems. Using the diffusion approximation technique, Jain M. [8] investigated a double-ended queue with generic arrival. Kimura [12] proposed a unifying diffusion approximation model for state-dependent queues. Garg et al. [6] used diffusion approximation to investigate the $G/G^Y/m$ queuing model with discouragement. Jain [9] later explained a diffusion approximation for the G/G/1 double-ended queue. Whit [24] showed a diffusion approximation for the $G/G^I/n/m$ Queue. If any big crowded program is organized, sometimes that causes many people to be harmed. Now that the pilgrims require quick medical attention, just a few doctors or servers are available, which is limited in real situations. On the other side, private hospitals have to wait for customers.

In this paper, $G_\infty/G_M/1$ “horizontal *Y*” shaped model has been considered in which there are bulk arrival patients but available services are limited by effective reflecting boundaries at *M* and *N*. Finally, develop the discrete distribution equation for queue size with various cases. Section 1, explains about introduction, motivation and about literature. Section 2, explained the proposed double-ended “horizontal *Y*” queue model, in which the first approximate solution, second approximate solution and normalizing condition have been derived. Section 3 deals with discrete distribution for queue size. Last Section 4 discussed the result.

Development of the proposed model

Assumed double-ended queue model with no limit of arrival patients in the system but occupying area for arrival patients have enclosed in which on *N* patients stay for *M* services and remaining are outside of this enclosed area and they will wait for service. When these first *N* patients will take services the remaining patients will come in an enclosed area, we can also understand with the help of Fig. 2.

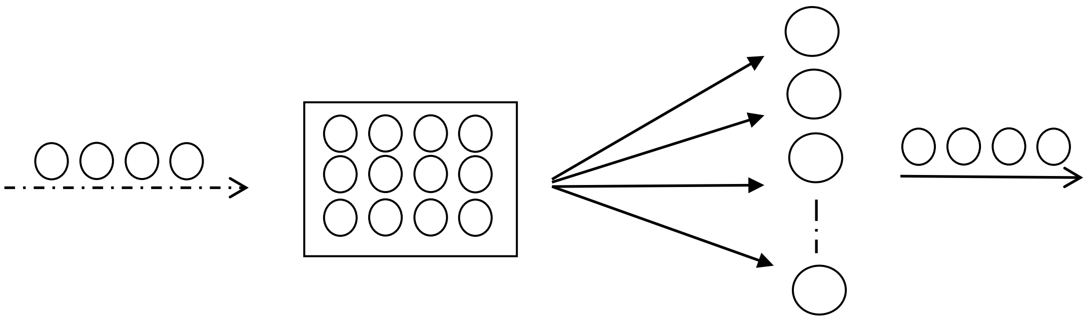

Fig. 2 A systematic queue structure

In inter-arrival times suppose patients/services are identically and independently distributed with mean arrival rate $\lambda(\mu)$ and square coefficient of variation P_{a^2}/P_{s^2}. The arrival patients baulk with probability $p = n/N$, where $n = 0, 1, 2, 3, \ldots, N$.

Suppose there are *n* customers (requests) in the queuing system at time *t* and it is denoted by:

$$Pn(t) = \begin{cases} \text{Prob \{patients waiting in outside and inside of the enclosed area at time } t\} \\ \text{Prob } \{-M \text{ services waiting for patients at time } t\} \end{cases}.$$

If $N(t)$ is the number of units present in the system at a time $t \geq 0$, then the diffusion approximation methodology dictates that we replace the discrete-valued variable $\{N(t), t \geq 0\}$ by the continuous-valued variable $\{X(t), t \geq 0\}$. Pandey and Gangeshwer [15] derived $\alpha(x)$ and $\beta(x)$ which is the infinitesimal mean and variance of the process. Using this $\alpha(x)$ and $\beta(x)$, integrating and using some properties of exponential and logarithmic functions.

Let $p(x)$ be the steady-state probability density function of $x(t)$:

$$P(x) = \frac{k}{\beta(x)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(2 \int_0^x \frac{\alpha(x)}{\beta(x)} \mathrm{d}x\right)^n. \tag{1}$$

First approximate solution

The solution $P_1(x)$ of Eq. (1) when $x < 0$ becomes

$$P_1(x) = \frac{k}{\lambda P_{a^2} + \mu P_{c^2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{2 \int_0^x \frac{\lambda - \mu}{\lambda P_{a^2} + \mu P_{c^2}} \mathrm{d}x\right\}^n,$$

$$P_1(x) = \frac{k}{R} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{2D \int_0^x \mathrm{d}x\right\}^n,$$

where $D = \frac{(\lambda - \mu)}{\lambda P_{a^2} + \mu P_{c^2}}$ and $R = \lambda P_{a^2} + \mu P_{c^2}$,

$$P_1(x) = (-1)^n K g_1(x). \tag{2}$$

Here $g_1(x) = \frac{1}{R}\sum_{n=0}^{\infty}\frac{1}{n!}(2Dx)^n$,

where $g_1(x)$ is continuous at $x = 0$ and $g_1(0) = \frac{1}{R}$.

Seared approximate solution

Now, the solution $P_2(x)$ of Eq. (1) when $x \geq 0$ become

$$P_2(x) = \frac{K}{\lambda P_{a^2}(1-x/N)+\mu P_{c^2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left\{2\int_0^x \frac{\lambda\left(1-\frac{x}{N}\right)-\mu}{\lambda P_{a^2}\left(1-\frac{x}{N}\right)+\mu P_{c^2}}dx\right\}^n.$$

Changing improper fractions to proper fractions and after integration we obtain:

$$P_2(x) = \frac{NK}{N\lambda P_{a^2}-x\lambda P_{a^2}+\mu N P_{s^2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{2x}{P_{a^2}}+\frac{2\mu N}{\lambda P_{a^2}}\left(1+\frac{P_{s^2}}{P_{a^2}}\right)\left\{log\left(1-\frac{x\lambda P_{a^2}}{\mu N P_{s^2}+\lambda N P_{a^2}}\right)\right\}\right]^n,$$

$$P_2(x) = \frac{NK}{N\lambda P_{a^2}-x\lambda P_{a^2}+\mu N P_{s^2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{2x}{P_{a^2}}\left\{\frac{N\mu P_{s^2}+\lambda N P_{a^2}-x\lambda P_{a^2}}{N\mu P_{s^2}+\lambda N P_{a^2}}\right\}^{\frac{2\mu N}{\lambda P_{a^2}}}\right]^n,$$

$$P_2(x) = \frac{K}{\lambda P_{a^2}+\mu P_{c^2}}\left\{1-\frac{x}{N\left(1+\frac{\mu P_{s^2}}{\lambda P_{a^2}}\right)}\right\}^{\frac{2\mu N}{\lambda P_{a^2}}\left(1+\frac{P_{s^2}}{P_{a^2}}\right)-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2x}{P_{a^2}}\right)^n,$$

$$P_2(x) = \frac{K}{R}\left\{1-\frac{x}{A}\right\}^{B-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2x}{P_{a^2}}\right)^n,$$

where $A = N\left(1+\frac{\mu P_{s^2}}{\lambda P_{a^2}}\right)$ and $B = \frac{2\mu N}{\lambda P_{a^2}}\left(1+\frac{P_{s^2}}{P_{a^2}}\right)$,

$$P_2(x) = \frac{K}{R}g_2(x), \tag{3}$$

where $g_2(x) = \left\{1-\frac{x}{A}\right\}^{B-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2x}{P_{a^2}}\right)^n$.

Here $g_2(x)$ is continuous at $x = 0$ and $g_2(0) = 1$.

Since both $g_1(x)$ and $g_2(x)$ are continous at point $x = 0$, we have:

$$\frac{g_1(0)}{g_2(0)} = \frac{1}{R}.$$

Normalizing condition

Using imposed reflecting boundaries $-M$ to N:

$$\int_{-M}^{0} P_1(x)dx + \int_{0}^{N} P_2(x)dx = 1,$$

$$\int_{-M}^{0} \frac{k}{R} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{2D \int_{0}^{x} dx\right\}^{n} dx + \int_{0}^{N} \frac{K}{R} \left\{1 - \frac{x}{A}\right\}^{B-1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2x}{P_{a^2}}\right)^{n} dx = 1,$$

$$K = \frac{2D\left(\lambda P_{a^2} + \mu P_{c^2}\right)}{\left\{\sum_{n=0}^{\infty} \frac{1}{n!}(2DM)^n - 1\right\} + 2DI_1}, \tag{4}$$

where

$$I_1 = \int_{0}^{N} \left\{1 - \frac{x}{A}\right\}^{B-1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2x}{P_{a^2}}\right)^{n} dx.$$

The above integral can be solved using the "incomplete Gamma function":

$$\gamma(z, \alpha) = \int_{0}^{\alpha} t^{z-1} e^{-t} dt.$$

We get $I_1 = L\left[\gamma\left(B, \frac{2a}{P_{a^2}}\right) - \gamma\left(B, \frac{2(A-N)}{P_{a^2}}\right)\right]$.

Finally,

$$K = \frac{R}{\frac{1}{2D}\left\{\sum_{n=0}^{\infty} \frac{1}{n!}(2DM)^n - 1\right\} + L\left[\gamma\left(B, \frac{2a}{P_{a^2}}\right) - \gamma\left\{B, \frac{2(A-N)}{P_{a^2}}\right\}\right]}.$$

Results of the proposed model

The discrete distribution for queue size is derived in the following manner:

$$P(n) = \begin{cases} \frac{(-1)^n K}{R} \sum_{n=0}^{\infty} \frac{1}{n!} (2Dx)^n, & n < 0. \\ \left\{1 - \frac{x}{A}\right\}^{B-1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2x}{P_{a^2}}\right)^{n}, & n \geq 0. \end{cases}$$

$P(n) = \int_{n-\frac{1}{4}}^{n+\frac{1}{4}} P(x)dx$ when $n < 0$.

Then $P(n) = \frac{K}{2DR}\left(\sum_{n=0}^{\infty} \frac{D^n}{n!} - 1\right) \sum_{n=0}^{\infty} \frac{\left\{-2D\left(n+\frac{1}{4}\right)\right\}^n}{n!}$.

If $n \geq 0$, then $P(n) = KL\left[\gamma\left(\mathrm{B}, \frac{2\left(\mathrm{A-n+\frac{1}{4}}\right)}{\mathrm{P_{a^2}}}\right) - \gamma\left\{\mathrm{B}, \frac{2\left(\mathrm{A-n-\frac{1}{4}}\right)}{\mathrm{P_{a^2}}}\right\}\right]$.

Now, for another value:

$P(n) = \int_{n-\frac{3}{4}}^{n+\frac{3}{4}} P(x)dx$ if $n < 0$, then

$$P(n) = \frac{K}{2DR}\left(\sum_{n=0}^{\infty}\frac{(3D)^n}{n!} - 1\right)\sum_{n=0}^{\infty}\frac{\left\{-2D\left(n+\frac{3}{4}\right)\right\}^n}{n!} \ .$$

If $n \geq 0$, then $p_n = KL\left[\gamma\left(B,\ \frac{2\left(A-n+\frac{3}{4}\right)}{P_{a^2}}\right) - \gamma\left\{B,\ \frac{2\left(A-n-\frac{3}{4}\right)}{P_{a^2}}\right\}\right]$.

In the case of unlimited arrival of patients, the mean is denoted by L_c and evaluated by

$$L_c = \int_0^{\infty} x\left\{1-\frac{x}{A}\right\}^{B-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\mathrm{x}}{\mathrm{P_{a^2}}}\right)^{\mathrm{n}} dx.$$

The above equation can be solved using "Gamma Function" and "incomplete Gamma Function" $\gamma(z+1\,,\alpha) = z\,\gamma(z\,,\alpha) - \alpha^z e^{-\alpha}$.

Then the mean value of the patient's queue length

$$L_c = \left[\left(\frac{BP_{a^2}}{2A} - 1\right)\left\{\Gamma B - \gamma\left(B, \frac{2A}{P_{a^2}}\right)\right\} + \left(\frac{2A}{P_{a^2}}\right)^{B-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{2A}{P_{a^2}}\right)^n\right].$$

The mean value of the queue length is denoted by L_s and calculated by

$$L_s = \int_{-M}^{-1} x\ \frac{(-1)^n K}{R}\sum_{n=0}^{\infty}\frac{1}{n!}(2Dx)^n\, dx.$$

Finally, we get

$$L_s = \frac{K}{4RD^2}\left[(2D-1)\sum_{n=0}^{\infty}\frac{1}{n!}(2D)^n + (1-2DM)\ \sum_{n=0}^{\infty}\frac{1}{n!}(2DM)^n\right].$$

Discussion and conclusion

The diffusion approximation is very helpful for discrete distribution for queue size in various cases. The diffusion approximation can be used to predict performance characteristics such as queue length and waiting time distributions. This paper discusses the "horizontal *Y*" shaped queuing model. The mean L_C (length of the customer) for arrival patients and mean L_S (length of the system) for finite servers have been derived. Finally, healthcare models created to assist patients' service systems might be considered for the possible further real-world applications. The approach developed in this paper is powerful and could be used to analyse more complex queuing systems.

Jain [10] also supports the application of a double-ended queue such that clients are arriving in groups of varying sizes or idle servers serving in batches. Arriving clients form a service queue, while an idle server is a taxi, and waits for consumers which are passengers. There is scope for the researcher to study queuing theory considering healthcare institutions. The present development would be useful for queuing scientists and hospital infrastructure construction.

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